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SKETCH

OF

THE CONSTRUCTION OF THIS WORK,

AND

CONTENTS OF PART FIRST.*

For the convenience of reference, these Mechanics of Fluids are distributed into a series of chapters, whose titles indicate the several topics that receive mathematical demonstration. The first of these contains, besides a few brief but necessary definitions, the fundamental proposition upon which all the problems that are drawn up in this part of the work are in reality founded.

The principle established in the general proposition, enables the reader to proceed in the second chapter with the pressure of incompressible fluids upon physical lines, rectangular parallelograms considered as independent planes immersed in the fluid, and to determine the position of the centre of gravity of the various rectangular figures which the successive problems embrace, together with the pressures of fluids upon the sides and bottoms of cubical vessels, with the limits which theory assigns to the requisite thickness of flood-gates.

One distinguishing characteristic in this inquiry is, that every problem is accompanied by a practical example; and in order that nothing be omitted which could render the subject intelligible to the general reader, the most important formulæ of a practical and general nature have been thrown into rules, in words at length, whereby all the arithmetical operations required in the solution of the examples can be performed without any reference to the algebraical investigation, which is the surest way of uniting precept with example.

After the same method, the third chapter treats of the pressure exerted by non-elastic fluids upon parabolic planes immersed in these fluids, with the method of finding the centre of gravity of the space included between any rectangular parallelogram and its inscribed parabolic plane. This is a valuable proposition in the practice of bridge building, and it will therefore be more fully expounded in Hydraulic Architecture. It is very satisfactory to find, in prosecuting one branch of science, the means of accomplishing another; to discover in a subject purely hydrostatic, a method by which to find the position of the centre of gravity of the arch, with all its balancing materials, and consequently many important particulars respecting the weight and mechanical thrust, with the thickness of the piers necessary to resist the drift or shoot of a given arch, independently of the aid afforded by the other arches. The method laid down in Problem XII. for this purpose is presumed to be new; at any rate it has not been noticed by any previous writer on Mechanics. But its development is reserved for Hydraulic Architecture: the principle here established being all that is required in Hydrostatics.

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* With the last part of Vol. I. will be given its preface and table of contents.

Chapter IV. introduces the reader to the pressure of non-elastic fluids on circular planes and spheres immersed in those fluids, as independent bodies, the extremity of the diameter of the figure being in each case coincident with the surface of the fluid. These problems could easily have been extended to examples of elliptical planes and solids, but the investigation would not embrace any practical result: and where that is unattainable, this work presumes not to advance.

The Fifth Chapter, in which are classed the tetrahedron, cylinder, conical frustum, and indeed the frustum of any other regular pyramid, completes this branch of fluid pressure; but the investigation is directed altogether to the pressure of the fluid upon the internal surfaces of the vessels under consideration. Indeed this was part of the inquiry when the sphere was treated of in the fourth chapter; but in the fifth, the subject is purely practical, and involves some of the most important principles in the whole range of Hydrodynamics. The reader now enters upon that remarkable and important principle,

That any quantity of fluid, however small, may be made to balance or hold in equilibrio any other quantity, however great;

and is enabled thence to investigate the theory and expound the construction of those mechanical contrivances known as the hydrostatic press, bellows, and weighing machine, which are all methods of balancing different intensities of force, by applying the simple power of non-elastic fluids to parts of an apparatus moving with different velocities: and this is all the mechanical powers can effect.

The Sixth Chapter, which treats of these hydrostatic engines, their theory of construction and scientific description, commences with a distinct proposition; the first having proved sufficient to resolve every problem connected with fluid pressure upon rectilinear and curvilinear figures considered as independent planes immersed in the fluids, together with the pressure of fluids upon the interior surfaces of vessels containing the fluids and belonging to the class of regular bodies. The second proposition, which the reader now enters upon, involves the principle whereon depend the construction and appliancy of the hydrostatic press, an engine very generally employed, and which should therefore be scientifically as it is practically known. But the same proposition extends to the investigation of the hydrostatic bellows, and furnishes the principle of a new machine, by which goods may be weighed as by the common balance. It may thence be inferred, that as yet, science has but stepped on the threshold of fluids that are heavy and liquid. How far this distinguishing property—the power of transmitting pressure equally in all directions—may yet carry mankind, it would be idle to conjecture. Enough, however, is here shown to satisfy the reader, that in expounding the laws of the pressure and equilibrium of fluids, as well as those of their motion and resistance, he will encounter principles of great practical utility in the construction and use of machines, engines, apparatus, and instruments employed, not only in the higher departments of natural philosophy, but in the every day concerns of society, in the arts, manufactures, and domestic operations of civilized men. The occurrence of such principles seems to present the legitimate time and place for classifying the inventions to which they gave existence, and for directing genius in its attempts to elicit new applications of collateral principles; for though fortuitous circumstances and accidental hints may have led to some discoveries in hydrodynamics, the greater part of modern improvements must be traced to patient induction, which arrives at those coincidences whereby scientific men are enabled to expound the theory of particular machines, whose construction and principles of action depend upon the equilibrium or motion of fluids. By this method, nothing is taken for granted which can be investigated from a series of mathematical truths; for, as Mr. Whitehurst observes, it is one thing to assent to truths, and another to prove them to be true: the former leaves the mind in a state of suspense, the latter in the possession of truth.[*]

This Part concludes with some experiments upon the *quaqua versus* property of non-elastic fluids; these experiments have the lowly merit of placing that property—the power of transmitting pressure equally in all directions—in a popular point of view, "*level to the capacity of ordinary minds.*"

Such then is the plan upon which these Mechanics of Fluids are constructed; and such is the method proposed to be followed in treating of its subsequent *divisions.*

THE EDITOR.

[*] "Inquiry into the Original State and Formation of the Earth."—London, 1792.

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MECHANICS OF FLUIDS.

CHAPTER I.

DEFINITIONS AND OBVIOUS PROPERTIES OF WATERY FLUIDS, WITH THE PRELIMINARY ELEMENTARY PRINCIPLES OF HYDRODYNAMICS, FOR ESTIMATING THE PRESSURE OF INCOMPRESSIBLE FLUIDS.

1. THE phenomena of Hydrodynamics are those truths which explain the peculiarities of equilibrium and motion among fluid bodies, especially those that are heavy and liquid. As that branch of natural philosophy which points out and explains the properties and affections of fluids at rest, it comprehends the doctrine of pressure, specific gravity, equilibrium, together with the circumstances attending the positions, equilibrium, and stability of floating bodies, the phenomena of cohesion and capillary attraction. And as that other branch of natural philosophy which points out and explains the motions of such fluids as have weight and are liquids, it investigates the means by which such motions are produced, the laws by which they are regulated, the discharge of fluids through orifices of various dimensions, forms, and positions,—the motion of fluids in pipes, rivers, and canals, and the force or effect they exert against themselves, or against solid bodies which may oppose them. Hydrodynamics, therefore, from Greek words signifying *water* and *force*, comprehend the entire science of watery fluids, whether in a state of rest or of motion; and this science, practically considered, enables us to investigate and apply a fruitful source of maxims and principles, upon which depend the construction and efficiency of engines and machines employed in the arts, manufactures, and domestic concerns of society, together with that extensive class of mechanical combinations displayed in the more delicate and important operations of HYDRAULIC ARCHITECTURE.

2. *A Fluid* is a body so constituted, that its parts are all ready to yield to the action of the smallest force or pressure, in whatsoever direction it may be exerted. The following are some of the simplest and most obvious properties of fluids.*

3. Every particle of a fluid presses equally in all directions, whether it be upwards or downwards, laterally or obliquely; consequently, the lateral pressure of a fluid is equal to its perpendicular pressure. The converse of this is equally obvious, and is thus expressed.

4. Every particle of a fluid in a state of quiescence, is pressed equally in all directions.

5. When a fluid is in a state of rest, the pressure exerted against the surface of the vessel which contains it, is perpendicular to that surface.

6. When a mass of fluid is in a state of rest, its surface is horizontal, or perpendicular to the direction of gravity.

7. If two fluids which do not mix, are poured into the same vessel, and suffered to subside, their common surface is parallel to the horizon; consequently the surfaces of fluids continue horizontal, when subjected to the pressure of the atmosphere.

8. The particles of a fluid, situated at the same perpendicular depth below the surface, are equally pressed.

9. When a fluid is in a state of rest, the pressure upon any of its constituent elements, wheresoever situated, is equal to the weight of a column of fluid particles, whose length is equal to the perpendicular depth of the particle or element pressed; consequently, the pressure on any particle varies in its perpendicular depth, and in any vessel containing a fluid in a state of rest, the parts that are deepest sustain the greatest pressure.

These principles, which flow immediately from the conditions of fluidity, are too simple and obvious to require demonstration, yet nevertheless, the writers on hydrostatical science generally accompany them with a sort of popular proof, which may be found in almost every treatise that has appeared on the subject. But our immediate object being to unfold the more important elementary principles, by the resolution of a series of examples dependent upon one general proposition, we have thought it unnecessary to exhibit the demonstrations here *(Note* A). The general proposition is as follows :—

---

* Fluids are generally divided into two sorts, *compressible* and *incompressible*, or *elastic* and *non-elastic*; the latter of which, or *incompressible* and *non-elastic fluids*, such as water, mercury, wine, &c., form the subject of the present article; the discussion of the *compressible* and *elastic fluids*, is reserved for another place. The compressibility of water is so small, that in all practical operations in mechanics its bulk or mass may generally be considered unalterable : for at a thousand fathoms depth it can only be compressed one-twentieth of its bulk at the surface.

PROPOSITION I.

10. When an incompressible and non-elastic fluid is in a state of equilibrium, and subjected only to the action of gravity:—

> *The magnitude, or the intensity of pressure exerted by the fluid, perpendicularly to any surface immersed in it, or otherwise exposed to its influence, is measured by the weight of a column of the fluid, whose base is equal to the area pressed, and whose altitude is the same as the depth of the centre of gravity of that area beneath the upper surface of the fluid.*

This is an elegant and most important proposition in the doctrine of fluid pressure, and in order that the principle may be the more readily perceived, and the demonstration the more easily comprehended, it will be proper, in the first place, to exhibit and demonstrate an analogous property, in reference to the common centre of gravity of a system of bodies, or of the particles of matter of which the system is composed.

The property which we have alluded to above, is noticed by almost every writer on the principles of mechanical science, and it has at various times received most beautiful and rigorous demonstrations; it may therefore, at first sight, appear superfluous to introduce it here; but in order to bring the subject more immediately before the attention of our readers, we do not hesitate to repeat the process.

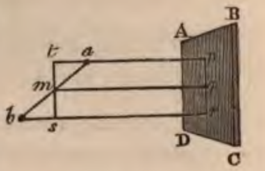
PROPOSITION (A).

11. If there be any system of bodies and a plane given in position with respect to them:—

> *The distance of that plane from the common centre of gravity of the system, is equal to the aggregate of the products, arising from multiplying each body into its distance from the given plane, divided by the sum of the bodies.*

The proposition just enunciated, being of the greatest use in many departments of philosophical inquiry, and of essential importance in establishing the truth of the hydrodynamic principle above specified, we shall therefore bestow some attention on its illustration for the purpose of rendering it as clear as possible, by connecting the steps with separate diagrams, and pursuing the reasoning, until we shall have proceeded so far that the law of induction becomes manifest, and from thence, the truth of the principle announced in the proposition.

To accomplish this purpose, let a and b, be two very small bodies



or particles of matter, supposed to be collected into their respective centres of gravity, and let A B C D be a smooth rectangular plane or surface, placed in any position with respect to the bodies a and b.

Connect a and b by the straight line ab, and let m be the place of their common centre of gravity; draw the straight lines $a\,p$, $m\,q$ and $b\,r$ respectively perpendicular to the plane A B C D, and consequently parallel to one another; join pr, then because the points a, m, b are situated in a straight line, the points p, q, r are also in a straight line, and therefore $p\,r$ will pass through the point q.

Through m, the common centre of gravity of the two bodies a and b, draw $s\,t$ parallel to pr, meeting br in s, and $p\,a$ produced in t; then the triangles amt and bms, are similar to one another; but by the property of the lever, we have

$$a : b :: bm : am,$$

and by similar triangles, it is

$$bm : am :: bs : at;$$

therefore, by the equality of ratios, we obtain

$$a : b :: bs : at;$$

from which, by equating the products of the extreme and mean terms, we get

$$a \times at = b \times bs.$$

Now, it is manifest by the construction, that $at = pt - pa$, and $b\,s = r\,b - r\,s$; therefore, by substitution, we obtain

$$a\,(p\,t - p\,a) = b\,(r\,b - r\,s);$$

but by reason of the parallels $p\,r$ and $t\,s$, the lines $p\,t$ and $r\,s$ are respectively equal to $m\,q$; hence we have

$$a\,(m\,q - p\,a) = b\,(r\,b - m\,q),$$

and from this, by collecting the terms and transposing, we get

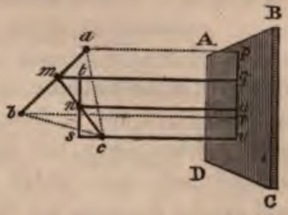
$$(a + b)\,m\,q = a \times p\,a + b \times r\,b,$$

and finally, by division, we obtain

$$m\,q = \frac{a \times p\,a + b \times r\,b}{a + b}.$$

Corol. Here then, the truth of the proposition is manifest with respect to a system composed of only two bodies; that is,

> *The distance of the common centre of gravity from the plane to which the bodies are referred, is equal to the sum of the products, arising by multiplying each body into its distance from the given plane, divided by the sum of the bodies.*



12. Again, let a, b and c, be a system of three very small bodies or particles of matter, any how situated with respect to the plane A B C D, and connected together by the straight lines $a b$, $b c$ and $a c$; and suppose the two bodies a and b to be collected into their common centre of gravity at the point m.

Join the points m and c by the straight line mc, and let n be the place of the common centre of gravity of the three bodies a, b and c; draw the lines mq, nu and cv parallel to each other, and respectively perpendicular to the plane A B C D; join qv, and because the points m, n and c are situated in the straight line mc; it follows, that the points q, u and v must also occur in a straight line; consequently, qv will pass through the point u.

Through n, the common centre of gravity of the three bodies a, b and c, draw st parallel to qv, meeting mq in t and vc produced in s; then are the triangles mnt and cns similar to one another; but by the property of the lever, and because the body at m is equal to $a+b$, we obtain

$$a+b : c :: cn : mn,$$

and by similar triangles, we have

$$cn : mn :: cs : mt;$$

therefore, by the equality of ratios, we get

$$a+b : c :: cs : mt;$$

consequently, by equating the products of the extreme and mean terms, we shall obtain

$$(a+b) \times mt = c \times cs;$$

now $mt = mq - tq$, and $cs = vs - vc$; hence we get

$$(a+b)(mq - tq) = c\,(vs - vc).$$

But it is manifest by the construction, that tq and vs are each of them equal to nu; therefore, by substitution we have

$$(a + b\,(mq - nu) = c\,(nu - vc);$$

therefore, by collecting the terms and transposing, we get

$$(a+b+c) \times nu = (a+b) \times mq + c \times vc;$$

now, it has already been shown in the case of two bodies, that

$$mq = \frac{a \times pa + b \times rb}{a+b};$$

therefore, by substituting this value of mq in the step immediately preceding, we shall obtain

$$(a+b+c) \times nu = a \times pa + b \times rb + c \times vc,$$

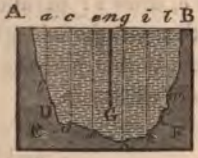
and finally by division, we have

$$nu = \frac{a \times pa + b \times rb + c \times vc}{(a+b+c)}.$$

Now, nu is the distance of the common centre of gravity of the three bodies a, b and c, from A B C D the plane to which they are referred; hence again, the truth of the proposition is manifest, and if another body were added to the system, a similar investigation would exhibit the same law, and thus we might proceed to any extent at pleasure, the nature of the induction being fully disclosed.

Corol. If therefore, we suppose the system to be constituted of an indefinite number of small bodies or particles of matter, it will become assimilated to a fluid mass, and consequently, the proposition which we have just demonstrated in reference to the centre of gravity, is identified with the well-known theorem for estimating the pressure of fluids; to which subject we must now return.

13. Resuming therefore, the conditions specified in Proposition I preceding, let us suppose that ABCD, denotes a vertical section of a reservoir full of water, E and F representing the corresponding sections of the walls or embankments by which it is contained; then, since the fluid is supposed to be quiescent or in a state of equilibrium, it follows, that the surface A B is parallel to the horizon.



Let $b\,d\,f\,h\,k\,m$ be the portion of the containing section or boundary, on which the pressure exerted by the water is required to be investigated, and conceive it to be constituted of an indefinite number of minute bodies or particles of matter, placed at infinitely small distances from one another, or so near, that their aggregate or sum shall make up the entire area which forms the subject of our investigation.

Suppose the points b, d, f, h, k and m, to be so many individual particles of the surface pressed, and through the points thus assumed draw the vertical lines $b\,a$, $d\,c$, $f\,e$, $h\,g$, $k\,i$ and $m\,l$, which lines are severally in the direction of gravity, and consequently perpendicular to the surface of the fluid, indicating by their lengths, the respective depths of the several bodies of which our immediate system is composed.

But according to Proposition I, the pressures exerted by the fluid on the particles b, d, f, h, k and m, are respectively represented by the products

$b \times ba$, $d \times dc$, $f \times fe$, $h \times hg$, $k \times ki$ and $m \times ml$, and the aggregate or sum of these products becomes

$$p = b \times ba + d \times dc + f \times fe + h \times hg + k \times ki + m \times ml,$$

where p denotes the sum of the computed pressures.

Now, it is manifest, from what we have demonstrated in Proposition (A), respecting the centre of gravity of a system of bodies, that

> *The sum of the products, arising from multiplying each body into its distance from a certain plane given in position, is equal to the sum of the bodies, drawn into the distance of their common centre of gravity from that plane.*

Let therefore, the particles b, d, f, h, k and m be considered as a system of very minute bodies, and let the surface of the fluid denote the plane given in position, to which the system is referred; then, if G be the place of the common centre of gravity of that system, put $n\text{G} = \delta$, and we shall obtain

$$\delta\,(b+d+f+h+k+m) = b\,.\,ba + d.dc + f\,.\,fe + h\,.\,hg + k\,.\,ki + m.ml.$$

But we have seen above, that the sum of the products on the right hand side of the equation, expresses the aggregate pressure on the several points of the containing surface, to which the present step of the inquiry refers, and that pressure we have briefly represented by the symbol p; therefore we have

$$p = \delta\,(b+d+f+h+k+m),$$

and this expression implies, that the pressure exerted by a fluid, on any number of points of the surface that contains it,

> *Is equal to the sum of the points, drawn into the perpendicular distance of their common centre of gravity below the upper surface of the fluid.*

Now, it is evident, that the same law would obtain if another point were added to the system, and even if the number of points were to become indefinite, or such that their aggregate or sum shall be essentially equal to the area pressed, the law of induction would remain the same; consequently, if a denote the sum of the material points, or particles of space in the surface on which the fluid presses; then we shall have

$$p = \delta a. \qquad (1).$$

This equation supposes, that the specific gravity of the fluid by which the pressure is propagated, is represented by unity, which circumstance only holds in the case of water; therefore, let s denote the specific gravity of any incompressible fluid whatever, and the general form of the equation becomes

$$p = \delta a s. \qquad (2).$$

Now, it is obvious, that the expression $\delta a s$ indicates the weight of a column of the fluid, the area of whose base is a, perpendicular altitude δ, and the specific gravity s; hence the truth of the proposition is manifest.

COROL. From what has been demonstrated above, it appears, that whatever may be the form of the surface on which the fluid presses, if its area, and the position of its centre of gravity can be ascertained, the intensity of pressure which it sustains, is from thence assignable.

The truth of the proposition being thus established, we shall proceed to deduce from it a few of the most useful and obvious inferences.

14. INF. 1. If different planes be immersed perpendicularly, horizontally, or obliquely, in fluids of different specific gravities :—

> *The pressures upon those planes perpendicularly to their surfaces, are as their areas, the perpendicular depths of their centres of gravity, and the specific gravities of the fluids jointly.*

15. INF. 2. If different planes be immersed perpendicularly, horizontally, or obliquely in the same fluid :—

> *The pressures upon those planes perpendicularly to their surfaces, are as their areas, and the perpendicular depths of their centres of gravity.*

16. INF. 3. If a plane surface of given dimensions be parallel to the surface of the fluid in which it is immersed :—

> *The pressure sustained by the plane, in a direction perpendicular to its surface, varies directly as its vertical depth below the upper surface of the fluid.*

17. INF. 4. If a plane surface of given dimensions be any how inclined to the surface of the fluid in which it is immersed :—

> *The pressure sustained by the plane, in a direction perpendicular to its surface, varies directly as the vertical depth of its centre of gravity, below the upper surface of the fluid.*

18. INF. 5. If any number of planes of equal areas be immersed in the same fluid, and have their centres of gravity at the same vertical depth below the surface :—

> *The pressures which they sustain are equal to one another, whatever be their form, and whatever be their position with respect to the surface of the fluid.*

19. INF. 6. If any plane surface revolve about its centre of gravity, which remains fixed in position :—

The pressure which it sustains in a direction perpendicular to its surface, will be the same at every point of the revolution as if it remained constantly horizontal.

20. INF. 7. If the perpendicular pressures upon a given surface be equal, when it is immersed in two fluids of different densities:—

The perpendicular depths of the centres of gravity below the surface, will vary inversely[*] *as the densities or specific gravities of the fluids.*

21. The above inferences are immediately deducible from the general proposition, but it is probable that the last may require a little illustration; for which purpose—

Put $p =$ the pressure sustained by the plane in both the fluids,
$a =$ the area of the plane or the surface pressed,
$s =$ the density or specific gravity of one of the fluids,
$d =$ the depth at which the given surface is immersed in it,
$s' =$ the density or specific gravity of the other fluid,
and $\delta =$ the depth of immersion.

Then, according to the principle indicated by the general equation (2), we have, in the case of the first fluid,

$$p = d\,a\,s,$$

and in the case of the second fluid, it is

$$p = \delta\,a\,s';$$

but according to the conditions of the question, these expressions are equal to one another, for the pressure is the same in both cases; consequently by comparison, we have

$$d\,a\,s = \delta\,a\,s',$$

and this, by suppressing the common factor, becomes

$$d\,s = \delta\,s';$$

therefore, by converting this equation into an analogy or proportion, we shall exhibit the precise conditions of the inference; hence, we have

$$d : \delta :: s' : s.$$

---

[*] One quantity is said to vary inversely as another, when of two quantities the one increases as the other decreases.

CHAPTER II.

OF THE PRESSURE OF NON-ELASTIC FLUIDS UPON PHYSICAL LINES, RECTANGULAR PARALLELOGRAMS CONSIDERED AS INDEPENDENT PLANES IMMERSED IN THE FLUIDS, AND UPON THE SIDES AND BOTTOMS OF CUBICAL VESSELS, WITH THE LIMIT TO THE REQUISITE THICKNESS OF FLOODGATES.

1. OF THE PRESSURE OF FLUIDS ON PHYSICAL LINES.

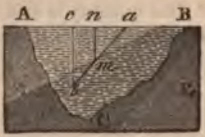
THE principle established in the general proposition enables us now to proceed with the resolution of a numerous class of curious and important problems, which will be found of the greatest practical utility, in all cases in which the pressure of watery fluids is concerned. These problems we shall accompany by examples, which will unfold their geometrical and analytical character, and leave no truth in the phenomena of this branch of hydrodynamics unrevealed.

PROBLEM I.

22. A physical line,* of a given length, is obliquely immersed in an incompressible fluid in a state of equilibrium, in such a manner that its upper extremity is just in contact with the surface;—

> *It is required to determine what pressure it sustains, the angle of obliquity being a given quantity.*

Let A B C, represent a vertical or upright section of a lake or pool of stagnant water, confined by the walls or embankments of which E E is a vertical section, and let A B be the surface of the water, supposed by the problem to be in a state of equilibrium.



In A B take any point a, and at the point a thus assumed, immerse the line $a\,b$ of the given length, and tending downwards at the given inclination or angle $b\,a$ A.

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* A *physical line* is that which belongs to, or exists in nature, and is so called to distinguish it from a mathematical line, which exists only in the imagination.

Bisect ab in m, and through the point m draw mn perpendicular to AB, the surface of the fluid; then, because the centre of gravity of a straight line is at the middle of its length, m is the place of the centre of gravity, and nm its perpendicular depth below the surface AB; through b draw the straight line bc parallel to mn, and cb is the perpendicular depth of the lower extremity at b.

Put $l = ab$, the length of the line whose upper extremity is at a,
$d = nm$, the perpendicular depth of the centre of gravity,
$\phi = bac$, the angle of inclination, or the given obliquity.

Then, because m is the centre of gravity of the straight line ab, we have $am = \frac{1}{2} l$, and by the principles of Plane Trigonometry, we obtain

$$\text{rad.} : \sin. \phi :: \tfrac{1}{2} l : d,$$

and since the tabular radius is expressed by unity, we get

$$d = \tfrac{1}{2} l \sin. \phi.$$

Now, the whole pressure which the line sustains in a direction perpendicular to its length, according to the second inference preceding,

> *Is proportional to its area, drawn into the perpendicular depth of its centre of gravity below the upper surface of the fluid.*

But the area of a physical line is simply equal to its length; therefore, if the symbol p denote the pressure, and s the specific gravity of the fluid by which it is propagated, we shall have

$$p = \tfrac{1}{2} s l^2 \sin. \phi. \qquad (3).$$

and this, in the case of water, where the specific gravity is expressed by unity, becomes

$$p = \tfrac{1}{2} l^2 \sin. \phi.$$

23. This equation, as well as the more general one from which it is derived, is sufficiently simple in its form for practical application; but in order that nothing may be omitted, which tends to render the subject intelligible to our readers, we shall in this, and in all the succeeding formulæ of a practical or general nature, draw up a rule, describing the manner in which the several steps of the process are to be performed; pursuant to this plan, therefore, the rule for the present case will be as follows:—

> RULE. *Multiply the square of the length by half the specific gravity of the fluid, and again by the natural sine of the angle of inclination, and the product will express the required pressure on the line in the oblique position.*

24. EXAMPLE 1. A physical line whose length is 36 feet, is immersed in a cistern of water, in such a manner that the upper extremity

is just in contact with the surface, and the other inclining downwards in an angle of 67° 35′; what pressure does the line sustain, supposing the fluid in which it is placed to be in a state of equilibrium?

Here by the question, the fluid in which the line is supposed to be immersed is water, of which the specific is unity; consequently, according to the rule, we have

$$p = 36 \times 36 \times \tfrac{1}{2} \times \sin. 67° 35';$$

but by the Trigonometrical Tables, the natural sine of 67° 35′ is .92444; hence we get

$$p = 1296 \times \tfrac{1}{2} \times .92444 = 599.03712.$$

In this case, however, the resulting pressure is only relative, the absolute pressure being indeterminable, upon a line where length merely is indicated and no breadth assigned; the existence of surface being indispensable for the expression of a determinate measure.

25. If the line were immersed perpendicularly in the fluid, or so as to make a right angle with its surface, the equation (3) would become transformed into

$$p = \tfrac{1}{2} s l^2 \sin. 90°;$$

but by the principles of Trigonometry, we have

$$\sin. 90° = 1;$$

hence, by substitution, we obtain

$$p = \tfrac{1}{2} s l^2; \qquad (4).$$

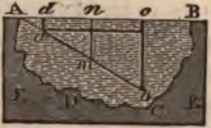
and this, in the case of water, where the specific gravity is unity, becomes

$$p = \tfrac{1}{2} l^2.$$

Therefore, the relative pressure for a perpendicular immersion, on the line, as given in the above example, is

$$p = 36 \times 36 \times \tfrac{1}{2} = 648.$$

26. If the upper extremity of the line be not in contact with the surface of the fluid, but placed as in the annexed diagram, then the method of solution, and consequently the form of the resulting equation, will be somewhat different.



Let A B be the surface of the water or fluid in which the line is immersed, and A B C D a vertical section, in whose plane the line $a\,b$ is situated, E E being the corresponding section of the walls or embankments by which the fluid is contained.

Bisect the given line ab in m, and through the point m thus determined, draw $m\,n$ perpendicular to A B, the surface of the fluid; and through a and b the extremities of the given line, and parallel to mn, draw ad and bc, and produce ba to meet A B in A, or in any other point, according to circumstances; then is mn the depth of the centre of gravity of the line ab, below the surface of the quiescent fluid, and ad, bc are respectively the depths of its extremities, $b\text{A}c$ being the angle which the direction of the given submerged line makes with the horizontal line A B.

Put $d = ad$, the depth of the upper extremity of the given line,
$\delta = mn$, the depth of the centre of gravity,
$\text{D} = bc$, the depth of the lower extremity,
$l = ab$, the length of the proposed line,
$p =$ the relative pressure upon it as propagated by the fluid,
and $\phi = b\text{A}c$, the angle which the given line makes with the horizon.

Through a the upper extremity of the given line, draw ae parallel to A B the surface of the fluid; then is the angle bae equal to the angle $b\text{A}c$, and by the principles of Plane Trigonometry, we have

$$ab : be :: \text{rad.} : \sin.\phi;$$

but be is manifestly equal to $bc - ad$; that is, $be = \text{D} - d$, and according to our notation, $ab = l$; hence, the above analogy becomes

$$l : (\text{D} - d) :: \text{rad.} : \sin.\varphi,$$

or by putting radius equal to unity, we get

$$\sin.\phi = \frac{(\text{D} - d)}{l}.$$

This equation enables us to determine the obliquity of the line, when the perpendicular depths of its two extremities are given; but when it is required to determine the relative pressure from the same data, we have only to observe, that mn the perpendicular depth of the centre of gravity, is equal to half the sum of the depths of the two extremities; that is,

$$\delta = \tfrac{1}{2}(\text{D} + d);$$

consequently, we obtain

$$p = \tfrac{1}{2}\, l\, (\text{D} + d).$$

Again, if the angle of inclination and the perpendicular depth of one extremity of the line are given, together with its length, the perpendicular depth of the other extremity can easily be found; thus, suppose that da is the given depth, then, by the principles of Plane Trigonometry, we have

$$be = l \sin.\phi;$$

but by addition, we obtain

$$bc = be + ec; \text{ that is, } \text{D} = l \sin.\phi + d;$$

consequently, the perpendicular depth of the centre of gravity, is

$$\delta = \tfrac{1}{2}\, l \sin.\phi + d;$$

and the relative pressure becomes

$$p = \tfrac{1}{2}\, l^2 \sin.\varphi\ \ ld.$$

But the equation, in its present form, supposes the specific gravity of the fluid to be expressed by unity, which only takes place in the case of water; in order, therefore, to generalize the formula, we must introduce the symbol which denotes the specific gravity; hence, we obtain

$$p = \tfrac{1}{2} l^2 s \sin. \phi + l s d;$$

or by collecting the terms, we get

$$p = l s (\tfrac{1}{2} l \sin. \phi + d). \qquad (5).$$

27. This is the general form of the equation, on the supposition that the perpendicular depth of the upper extremity of the line is given; it however assumes a different form, when the depth of the lower extremity is known; for by Plane Trigonometry, we have as above

$$b e = l \sin. \phi,$$

and by subtraction, we obtain

$$e c = b c - b e; \text{ that is, } d = \mathrm{D} - l \sin. \phi;$$

therefore, the perpendicular depth of the centre of gravity is

$$\delta = \mathrm{D} - \tfrac{1}{2} l \sin. \phi,$$

and consequently, the general expression for the pressure becomes

$$p = l s (\mathrm{D} - \tfrac{1}{2} l \sin. \phi). \qquad (6).$$

28. Therefore, the practical rule for each of these cases, when expressed in words at length, is as follows:—

1. When the perpendicular depth of the upper end is given (5).

   RULE. *To half the length of the given line drawn into the natural sine of the angle of inclination, add the depth of the upper extremity; then, multiply the sum by the length of the line, drawn into the specific gravity of the fluid, and the product will give the pressure sought.*

2. When the perpendicular depth of the lower end is given (6).

   RULE. *From the perpendicular depth of the lower extremity, subtract half the length of the given line drawn into the natural sine of the angle of inclination; then, multiply the remainder by the length of the line, drawn into the specific gravity of the fluid, for the pressure sought.*

29. EXAMPLE 2. A physical line, whose length is 56 feet, is immersed in a cistern of water, in such a manner that its upper extremity is at the distance of 9 feet below the surface, and its direction making with the horizon an angle of 58 degrees; required the relative pressure on the line, the water being in a state of quiescence?

The natural sine of 58 degrees, according to the Trigonometrical Tables, is .84805; therefore by the rule, we have

$28 \times .84805 + 9 = 32.7454$, the perpendicular depth of the centre of gravity; then, finally, because the specific gravity of water is unity, we have

$$p = 32.7454 \times 56 = 1833.7424.$$

Let the length of the line and its inclination to the horizon remain as above, and suppose the depth of the lower extremity to be 56.4908 feet; then, by the rule for the second case, we have

$56.4908 - 28 \times .84805 = 32.7454$, the depth of the centre of gravity, the same as above, from which the relative pressure is found to be 1833.7424, as it ought to be.

30. If the line were immersed perpendicularly, or at right angles to the horizon, then sin. ϕ is equal to unity, and the formulæ for the pressure become

$$p = l\,s\,(\tfrac{1}{2}\,l + d), \text{ and } p = l\,s\,(\text{D} - \tfrac{1}{2}\,l),$$

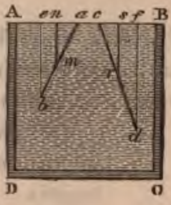
where it is manifest, that the parenthetical expressions are equal to one another, each of them expressing the perpendicular depth of the centre of gravity, or the middle point of the given line.

PROBLEM II.

31. Two physical lines of different given lengths, have their upper extremities in contact with the surface of an incompressible and non-elastic fluid in a state of equilibrium :—

> *It is required to compare the pressures which they sustain at right angles to their lengths, supposing them to be immersed at given inclinations to the horizon.*

Let A B C D, represent a vertical section of a vessel filled with water, or some other incompressible and non-elastic fluid, and suppose the lines $a\,b$ and $c\,d$ to be situated in the plane of the section, in such a manner that the upper extremities a and c are respectively in contact with A B the surface of the fluid, while their directions make with the horizon the angles $b\,a\,$A and $d\,c\,$B respectively.



Through the points b and d, the lower extremities of the lines ab and cd, draw be and df respectively perpendicular to A B the surface of the fluid; and through m and r, the middle points of $a\,b$ and $c\,d$, draw the lines mn and rs respectively parallel to the perpendiculars $b\,e$ and $d\,f$; then are $m\,n$ and $r\,s$ the perpendicular depths of the centres of gravity.

Put $l = a\,b$, the length of the line whose upper extremity is a,

$l' = c\,d$, the length of that whose upper extremity is c;

$d = n\,m$, the perpendicular depth of the centre of gravity of the line $a\,b$;

$\delta = s\,r$, the perpendicular depth of the centre of gravity of the line $e\,d$;
$\phi = b\,a\,\mathrm{A}$, the inclination of the line $a\,b$ to the horizon,
$\phi' = d\,c\,\mathrm{B}$, the inclination of $c\,d$ to the horizon, or to the line $\mathrm{A\,B}$;
$p =$ the relative pressure upon $a\,b$,
$p' =$ the relative pressure upon $c\,d$,
and $s =$ the specific gravity of the fluid.

Now, because the centre of gravity of a physical straight line is at the middle of its length, we have

$$a\,m = \tfrac{1}{2}l, \text{ and } c\,r = \tfrac{1}{2}l';$$

therefore, by the principles of Plane Trigonometry, we obtain from the right-angled triangle $a\,m\,n$

$$d = \tfrac{1}{2}\,l \sin.\,\phi,$$

and from the right-angled triangle $c\,r\,s$ we get

$$\delta = \tfrac{1}{2}\,l' \sin.\,\phi.$$

consequently, the general expressions for the relative pressures on the lines $a\,b$ and $c\,d$, according to equation (5) are

$$p = \tfrac{1}{2}\,l^2 s \sin.\,\phi, \text{ and } p' = \tfrac{1}{2}\,l'^2 s \sin.\,\phi',$$

from which, by comparison, we get

$$p : p' :: l^2 \sin.\,\phi : l'^2 \sin.\,\phi'.$$

INF. 1. Hence it appears, that the pressures on the lines, when their directions make different angles of inclination with the horizon,

Are directly as the squares of the lengths, and the sines of the inclinations jointly.

2. Where $\phi = \phi'$, that is, when the lines are equally inclined to the horizon, whatever may be the magnitude of the inclination, then

$$p : p' :: l^2 : l'^2;$$

therefore, when the lines are perpendicularly immersed, or when they are equally inclined to the surface of the fluid, with which their upper extremities are supposed to be in contact,

The pressures which they sustain perpendicular to their lengths, are directly proportional to the squares of those lengths.

3. Consequently, if two or more lines are similarly situated in the same fluid, the relative pressures can easily be compared; thus, for example:—

Suppose two physical lines, whose lengths are respectively 36 and 56 feet, to be perpendicularly immersed in the same fluid, and having their upper extremities in contact with the surface, or equally depressed below it; then, the pressures sustained by these lines, are to one another as the numbers 1296 and 3136; that is,

$$p : p' :: 36^2 : 56^2 :: 1296 : 3136.$$

But when the lines are differently situated in the fluid, the comparison of their relative pressures requires a more particular exemplification; for which purpose take the following example.

32. EXAMPLE 3. Two physical straight lines, whose lengths are respectively 18 and 27 feet, are immersed in the same fluid, in such a manner that their upper extremities are just in contact with its surface, and the angles which they make with the horizon are respectively equal to 42 and 29 degrees; what is the pressure on the longer line, supposing that on the shorter to be expressed by the number 78.54?

If we convert the preceding analogy for the oblique lines of different inclinations into an equation, by making the product of the mean terms equal to the product of the extremes, we shall obtain

$$p' l'^2 \sin.\phi = p\, l^2 \sin.\phi'.$$

Now, by assimilating the several quantities in this equation to the lines in the foregoing diagram, and according to the conditions of the question, it appears that p' is the required quantity, all the rest being given ; therefore, let both sides of the equation be divided by $l^2 \sin.\phi$, and we shall obtain

$$p' = \frac{p\, l'^2 \sin.\phi'}{l^2 \sin.\phi}.$$

But it is a well-known principle in the arithmetic of sines, that to divide by the sine of any arc, is equivalent to multiplying by the cosecant of that arc ; hence we have

$$p' = \frac{l'^2}{l^2}\,(p \sin.\phi' \operatorname{cosec}.\phi).$$

Let therefore the numerical values, as proposed in the example, be substituted for the respective symbols in the above equation, and we shall obtain

$$p' = \frac{27^2}{18^2}\,(78.54 \sin.29^\circ \operatorname{cosec}.42^\circ);$$

now, the natural sine of 29°, according to the Trigonometrical Tables, is .48481, and the natural cosecant of 42° is 1.49447 ; therefore, by substitution, we get

$$p' = \frac{27^2}{18^2} \times 78.54 \times .48481 \times 1.49447 = 128 \text{ nearly};$$

consequently the pressures on the inclined lines, are to one another as the numbers 78.54 and 128 ; but had the inclinations been equal, the comparative pressures would have been as 78.54 to 176.72 very nearly.

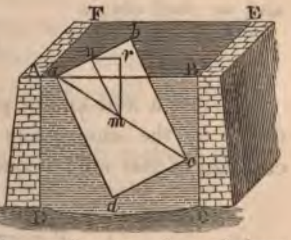
2. OF THE PRESSURE OF FLUIDS THAT ARE NON-ELASTIC UPON RIGHT ANGLED PARALLELOGRAMS CONSIDERED AS INDEPENDENT PLANES IMMERSED IN FLUIDS.

PROBLEM III.

33. A right angled parallelogram is immersed in a quiescent fluid, in such a manner, that one of its sides is coincident with the surface, and its plane inclined to the horizon in a given angle:—

> *It is required to determine the pressure perpendicular to the plane, both when it is inclined to the surface of the fluid, and when it is perpendicular to it, the nature of the fluid, and consequently its specific gravity, being known.*[*]

Let ABCD represent a vertical section of a volume of incompressible fluid in a state of equilibrium, of which ABEF is the surface, and consequently parallel to the horizon; let $a\,b\,c\,d$ be a rectangular plane immersed in the fluid, in such a manner that the upper side $a\,b$ coincides with the surface, and the plane $a\,b\,c\,d$ is inclined to the horizon in a given angle.



Draw the diagonal $a\,c$, which bisect in m, and through m the centre of gravity of the parallelogram, draw $m\,n$ parallel to ad or bc, meeting ab the line of common section perpendicularly in the point n.

In the horizontal plane ABEF, and through the point n, draw nr also at right angles to $a\,b$, and from m the centre of gravity of the immersed plane $a\,b\,c\,d$, let fall the perpendicular $m\,r$; then is the angle $m\,n\,r$ the inclination of the plane to the horizon, and $r\,m$ the perpendicular depth of its centre of gravity below the upper surface of the quiescent fluid.

Put $b = a\,b$, the horizontal breadth of the immersed parallelogram,
$l = a\,d$ or $b\,c$, the immersed length,
$d = r\,m$, the perpendicular depth of the centre of gravity,
$\phi = m\,n\,r$, the inclination of the plane to the horizon,
$p =$ the pressure on the plane perpendicularly to its surface,
and $s =$ the specific gravity of the fluid.

---

[*] By the pressure upon any plane or curvilineal surface, is always understood the aggregate of all the pressures upon every point of those surfaces, estimated in directions perpendicular to them at each point, no part being lost by obliquity of direction.

Then, because the point m is at the middle of ac, and mn parallel to ad, it follows, that $m\,n = \frac{1}{2}\,l$; and by reason of the right-angled triangle $m\,r\,n$, we have, from the principles of Plane Trigonometry,

$$r\,m = d = \tfrac{1}{2} l \sin.\phi;$$

consequently, the entire pressure upon the plane perpendicularly to its surface, is expressed by

$$p = \tfrac{1}{2}\, b\, l^2\, s \sin.\,\phi.$$

This is manifest from Problem I. (art. 22), for $\frac{1}{2} l \sin.\,\phi$ expresses the perpendicular depth of the centre of gravity, and $b\,l$ the area of the surface pressed; therefore, the solidity of the fluid column is

$$\tfrac{1}{2}\, l \sin.\,\phi \times b\,l = \tfrac{1}{2}\, b\, l^2 \sin.\,\phi,$$

and since s denotes the specific gravity of the fluid, the weight of the column is

$$\tfrac{1}{2}\, l \sin.\,\phi \times b\,l \times s = \tfrac{1}{2}\, b\, l^2\, s \sin.\,\phi;$$

but the perpendicular pressure upon the plane, is equal to the weight of the fluid column; therefore, we obtain

$$p = \tfrac{1}{2}\, b\, l^2\, s \sin.\,\phi. \qquad (7).$$

When the plane of the immersed rectangle is perpendicular to the surface of the fluid, we have $\phi = 90^\circ$, and $\sin.\phi = 1$; consequently, by substitution, the above equation becomes

$$p = \tfrac{1}{2}\, b\, l^2\, s. \qquad (8).$$

These equations are sufficiently simple in their form for practical application, and we shall show hereafter, that they are extremely useful in many important cases of hydrostatical construction.

34. The practical rules derived from these equations, for determining the pressure in the particular cases, may be expressed as follows.

1. When the plane is oblique to the horizon. (Eq. 7).

> Rule. *Multiply the square of the immersed length of the plane, by the horizontal breadth drawn into the specific gravity of the fluid, and again by the natural sine of the angle of inclination, and half the product will give the pressure sought.*

2. When the plane is perpendicular to the horizon. (Eq. 8).

> Rule. *Multiply the square of the immersed length of the plane, by the horizontal breadth drawn into the specific gravity of the fluid, and half the product will give the pressure sought.*

35. Example 4. A rectangular parallelogram, whose sides are respectively 18 and 3 feet, is immersed in a quiescent body of water, in such a manner, that its shorter side is in contact with the surface,

and its plane inclined to the horizon in an angle of 68 degrees; required the pressure which it sustains, both in the inclined and the perpendicular position?

In this example the area of the parallelogram is $18 \times 3 = 54$ square feet, and the longer side is that which is immersed downwards in the fluid; therefore, according to the rule for the oblique position, the solidity of the column by which the pressure is propagated, becomes

$$3 \times 18^2 \times s \times \tfrac{1}{2} \sin. 68^\circ = 486 \times s \sin. 68^\circ.$$

Now, in the case of water, the specific gravity is represented by unity, and by the Trigonometrical Tables, the natural sine of 68 degrees, is 0.92718; consequently, by substitution, the pressure becomes

$$p = 486 \times .92718 = 450.60948;$$

the pressure here obtained, however, is estimated in cubic feet of water; but in order to have it expressed in a more appropriate and definite measure, it becomes necessary to compare it with some weight; now, it has been found by experiment, that the weight of a cubic foot of water is very nearly equal to 62½ lbs. avoirdupois; therefore, the absolute pressure upon the plane, is

$$p = 450.60948 \times 62.5 = 28163.0925 \text{ lbs.}$$

36. Let the dimensions of the plane remain as in the preceding case, which condition is supposed in the example; then, the pressure on its surface, when perpendicular to the horizon, is

$$p = 3 \times 18 \times 18 \times 1 \times \tfrac{1}{2} = 486 \text{ cubic feet}$$

of water; but we have stated above, that the weight of one cubic foot is equal to 62½ lbs.; therefore, we have

$$p = 486 \times 62\tfrac{1}{2} = 30375 \text{ lbs.};$$

consequently, the pressures on the plane in the two positions, are to one another as the numbers 450.60948 and 486, when expressed in cubic feet of water; but when expressed in pounds avoirdupois, they are as the numbers 28163.0925 and 30375.

37. If the longer side of the rectangular parallelogram were coincident with the surface of the fluid, while its plane is obliquely inclined to the horizon; then, the formula for the pressure perpendicular to its surface becomes

$$p = \tfrac{1}{2} b^2 l s \sin. \phi. \qquad (9).$$

But if the plane of the parallelogram, instead of being inclined to the horizon, or which is the same thing, to the surface of the fluid, were immersed perpendicularly to it; then, $\phi = 90^\circ$, and $\sin. \phi = 1$; hence, the formula for the pressure becomes

$$p = \tfrac{1}{2} b^2 l s. \qquad (10).$$

Therefore, by retaining the data of the preceding example, the absolute pressure on the plane in the oblique position, is

$$p = 3^2 \times 18 \times 62.5 \times \tfrac{1}{2} \times .92718 = 4693.84875 \text{ lbs.}$$

But when the plane is perpendicularly immersed, the absolute pressure on its surface is

$$p = 3^2 \times 18 \times 62.5 \times \tfrac{1}{2} = 5062\tfrac{1}{2} \text{ lbs.}$$

Corol. 1. Hence, the pressures on the plane in the oblique and perpendicular positions, are to one another as the numbers 4693.84875 and $5062\frac{1}{2}$; but in order to compare the pressures under the same conditions, when the shorter and longer sides of the parallelogram are respectively in contact with the surface of the fluid, we have as follows, viz.

2. When the shorter side of the parallelogram is horizontal, the absolute pressure in the inclined position is 28163.0925 lbs.; but when the longer side is horizontal, the absolute pressure is 4693.84875 lbs.; consequently, the absolute pressures in the two cases are to one another as 6 to 1.

3. Again, when the shorter side of the parallelogram is horizontal, the pressure in the perpendicular position is 30375 lbs.; and when the longer side is horizontal, the pressure is $5062\frac{1}{2}$ lbs.; therefore, the pressures in these two cases are to one another as 6 to 1, the same as before; from which we infer, that the quantity of inclination affects only the magnitude of the pressures, and that in so far as it changes the position of the centre of gravity, but it has no effect upon the ratio; therefore, if the plane were to vibrate round its shorter and longer sides respectively as axes, the pressures on its surface, in the two cases, would be to one another in a constant ratio.

3. OF THE AGGREGATE PRESSURE EXERTED BY THE FLUID ON THE IMMERSED PARALLELOGRAM, AND ON EACH OF THE CONSTITUENT TRIANGLES FORMED BY ITS DIAGONAL.

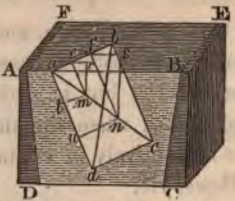
PROBLEM IV.

38. Suppose the parallelogram to be placed under the same circumstances as in the preceding problem, and let it be bisected by one of its diagonals :—

> *It is required to determine the aggregate pressure exerted by the fluid, in a direction perpendicular to the surface of each triangle into which the diagonal divides the parallelogram, and to compare the pressures on the two triangles.*

Let ABCD represent a vertical section of a mass or collection of quiescent fluid, contained by the walls or embankments indicated by the shaded boundary; and let ABEF be the horizontal surface of the fluid, with which one side of the immersed rectangle is supposed to be coincident.

Now, suppose $abdc$, to be the immersed rectangle, and draw the diagonal bd; then are abd and bdc the triangles, into which the parallelogram $abcd$ is divided by the diagonal bd, and for which the pressures are required to be investigated.



Draw the diagonal ac, which divide into three equal portions in the points m and n; then are m and n respectively the centres of gravity of the constituent triangles abd and bdc.

Through the points m and n, and parallel to ad or bc, the immersed sides of the figure, draw me and nf meeting ab perpendicularly in the points e and f; then, through the points e and f thus determined, and in the plane of the fluid surface, draw er and fs respectively perpendicular to ab; then are the angles mer and nfs equal to one another, and each of them is equal to the angle which the plane of the immersed parallelogram makes with the surface of the fluid.

From m and n, the centres of gravity of the triangles abd and bdc, demit the lines mr and ns respectively perpendicular to er and fs; then are rm and sn the perpendicular depths of the centres of gravity.

Put $b = ab$, the horizontal breadth of the immersed parallelogram,
$l = ad$ or bc, the immersed or downward length,
$d = rm$, the perpendicular depth of the centre of gravity of the triangle abd,
$\delta = sn$, the perpendicular depth of the centre of gravity of the triangle bdc,
$D = ac$ or ba, the diagonal of the parallelogram,
$\phi = mer$, or nfs, the inclination of the plane to the surface of the fluid,
$P =$ the whole pressure on the parallelogram $abcd$,
$p =$ the pressure on the triangle abd,
$p' =$ the pressure on the triangle bdc,
and $s =$ the specific gravity of the fluid.

Then, because the parallelogram $abcd$ is rectangular, the triangle *adc* is right angled at *d*; therefore, by the property of the right-angled triangle, we have

$$ac = \sqrt{ad^2 + dc^2};$$

or by employing the appropriate symbols, we have

$$\text{D} = \sqrt{l^2 + b^2}.$$

But, according to the construction, and by the nature of the centre of gravity, we have

$$am = \tfrac{1}{3} ac, \text{ and } an = \tfrac{2}{3} ac,$$

or symbolically, we obtain

$$am = \tfrac{1}{3}\sqrt{l^2 + b^2}, \text{ and } an = \tfrac{2}{3}\sqrt{l^2 + b^2}.$$

Now, by reason of the parallel lines *em*, *fn*, and *bc*, the triangles *aem*, *afn*, and *abc*, are similar among themselves; consequently, by the property of similar triangles, we have

$$ac : bc :: an : fn :: am : em;$$

therefore, by separating the analogies, and employing the symbols, it is

$$\text{D} : l :: \tfrac{2}{3}\sqrt{l^2 + b^2} : fn,$$

and again, we have

$$\text{D} : l :: \tfrac{1}{3}\sqrt{l^2 + b^2} : em;$$

from these analogies, therefore, we obtain $fn = \frac{2}{3}l$, and $em = \frac{1}{3}l$; which is otherwise manifest by drawing the dotted lines *mt* and *nu*.

Now, in the right angled triangles *erm* and *fsn*, there are given the hypothenuses *em* and *fn*, and the equal angles *mer* and *nfs*, to find *rm* and *sn*, the perpendicular depths of the centres of gravity; consequently, by Plane Trigonometry, we have, from the triangle *mer*,

$$\text{rad.} : \sin.\phi :: \tfrac{1}{3}l : d,$$

and from the triangle *nfs*, it is

$$\text{rad.} : \sin.\phi :: \tfrac{2}{3}l : \delta,$$

and since radius is equal to unity, these analogies become

$$d = \tfrac{1}{3}l \sin.\phi, \text{ and } \delta = \tfrac{2}{3}l \sin.\phi.$$

But according to Inf. 2, Proposition (A), the pressure sustained by each triangle, in a direction perpendicular to its surface,

> *Is expressed by the product of its area, drawn into the perpendicular depth of the centre of gravity.*

Now, the area of each triangle is manifestly equal to half the area of the given parallelogram, and by the principles of mensuration, the area of the rectangular parallelogram is equal to the product of its two dimensions; that is, of the length drawn into the breadth; therefore, we have for the pressure on the triangle *abd*,

$$p = \tfrac{1}{6} b l^2 \sin.\, \phi,$$

and in like manner, the pressure on the triangle *bdc* is

$$p' = \tfrac{1}{3} b l^2 \sin.\, \phi.$$

These equations, however, express the pressures simply by the magnitude of a fluid column, whose base is the area pressed, and whose altitude is equal to the depth of the centre of gravity below the upper surface of the fluid. In order, therefore, to have the pressures expressed in general terms, the specific gravity of the fluid must be taken into the account; in which case, the pressure on the triangle *abd* becomes

$$p = \tfrac{1}{6} b l^2 s \sin.\, \phi, \qquad (11).$$

and the pressure on the triangle *bdc* is

$$p' = \tfrac{1}{3} b l^2 s \sin.\, \phi. \qquad (12).$$

Corol. Hence it appears, that the pressure perpendicular to the plane of a triangle, when its vertex is upwards and coincident with the surface of the fluid, is double the pressure on the same triangle, when its base is upwards, and placed under the same circumstances.

39. If the immersed plane be perpendicular to the surface of the fluid, then $\phi = 90°$, and $\sin.\, \phi = 1$; therefore, by substitution, the preceding equations become

$$p = \tfrac{1}{6} b l^2 s, \text{ and } p' = \tfrac{1}{3} b l^2 s;$$

here again, the pressure in the one case is double the pressure in the other, and the same thing will obtain, whatever may be the inclination of the plane, provided only that *a b* coincides with the surface of the fluid; for then, the perpendicular depths of the centres of gravity will vary in a given ratio.

When the immersed plane is a square, that is, when *b* and *l* are equal to one another, the equations for the pressures in the oblique position become

$$p = \tfrac{1}{6} b^3 s \sin.\, \phi, \text{ and } p' = \tfrac{1}{3} b^3 s \sin.\, \phi,$$

and when the plane is perpendicular to the surface of the fluid, we have

$$p = \tfrac{1}{6} b^3 s, \text{ and } p' = \tfrac{1}{3} b^3 s.$$

Since the aggregate pressure upon the plane is equal to the sum of the pressures on the constituent triangles, the expression for the aggregate pressure in the oblique position, becomes in the case of a rectangle

$$P = p + p'; \text{ that is, } P = \tfrac{1}{6} b l^2 s \sin.\, \phi + \tfrac{1}{3} b l^2 s \sin.\, \phi = \tfrac{1}{2} b l^2 s \sin.\, \phi.$$

Corol. Hence it appears, that the pressures on the constituent triangles and that on the entire plane, are to one another as the numbers 1, 2 and 3; and the same thing obtains in the case of a square, whatever may be the inclination of the plane.

40. The practical rules for calculating the pressures on the triangles, as deduced from the equations (11) and (12) are as follows.

1. When the base of the triangle is coincident with the surface.

   RULE. *Multiply the square of the immersed length, or the perpendicular of the triangle, by the base drawn into the specific gravity of the fluid, and again by the natural sine of the plane's inclination, and one sixth part of the product will express the whole pressure upon the triangle.* (Eq. 11).

2. When the vertex of the triangle is coincident with the surface.

   RULE. *Multiply the square of the perpendicular of the triangle, by the base drawn into the specific gravity of the fluid, and again by the natural sine of the plane's inclination, and one third of the product will express the whole pressure on the triangle.* (Eq. 12).

41. EXAMPLE 5. A rectangular parallelogram, whose sides are respectively 26 and 14 feet, is immersed in a cistern of water, in such a manner, that its shorter side is coincident with the horizontal surface; what will be the pressure on each of the triangles, into which the parallelogram is divided by its diagonal, supposing its plane to be inclined to the surface of the fluid in an angle of 56° 35′?

Here, by the rule, we have

$$p = 26^2 \times 14 \times .83469 \times \tfrac{1}{6} = 1316.58436 \text{ cubic feet}$$

of water; but one cubic foot of water weighs 62½ lbs.; therefore, to express the pressure in lbs., we have

$$p = 1316.58436 \times 62\tfrac{1}{2} = 82286.5225 \text{ lbs.}$$

The pressure which we have just obtained, refers to that portion of the parallelogram which has its base coincident with the surface of the fluid; that is, to the triangle *abd*, and the pressure on the other portion, or the triangle *bdc*, is determined as follows.

$$p' = 26^2 \times 14 \times .83469 \times \tfrac{1}{3} = 2633.16872 \text{ cubic feet of water;}$$

or to express the pressure in lbs. we have

$$p' = 2633.1687 \times 62.5 = 164573.045 \text{ lbs.}$$

If the plane of the immersed parallelogram were perpendicular to the surface of the fluid, the pressures on the triangles *abd* and *bdc* would be respectively as follows.

$$p = 26^2 \times 14 \times 62\tfrac{1}{2} \times \tfrac{1}{6} = 98583\tfrac{1}{3} \text{ lbs., and } p' = 26^2 \times 14 \times 62\tfrac{1}{2} \times \tfrac{1}{3} = 197166\tfrac{2}{3} \text{ lbs.}$$

COROL. The circumstance of the aggregate pressure on the parallelogram, being equal to the sum of the pressures on the constituent triangles, furnishes a very simple and elegant method of determining the centre of gravity; which method, in so far as respects plane figures of particular forms, may, in many instances, be very advantageously applied.

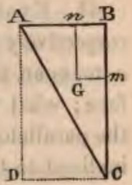
It would be foreign to our present purpose to enter into a detail of the method alluded to in this place; nevertheless, for the satisfaction of our readers, we shall briefly introduce it, not being aware that it has been suggested by any other writer in ancient or modern times.

PROBLEM B.

42. The base and perpendicular of a right angled triangle being supposed known:—

> *It is required to determine the position of its centre of gravity, or that point on which, if the surface were supported, it would remain at rest in any position.*

Let ABC be the triangle given, of which it is required to determine the centre of gravity.



Complete the parallelogram ABCD, by drawing the dotted lines AD and DC; then, because the entire pressure on the parallelogram ABCD, is equal to the sum of the pressures on the triangles ABC and ACD; it follows, that the pressure on the triangle ABC, is equal to the difference between the entire pressure on the parallelogram, and that on the triangle ACD; consequently, we have, by retaining the foregoing notation,

$$p = \mathrm{P} - p'; \text{ that is,}$$

$$p = \tfrac{1}{2} b l^2 - \tfrac{1}{3} b l^2 = \tfrac{1}{6} b l^2.$$

But it has been elsewhere demonstrated, that the pressure on any surface, is expressed by the area of that surface, drawn into the perpendicular depth of its centre of gravity; consequently, the perpendicular depth of the centre of gravity, must be equal to the pressure divided by the area of the surface.

Now, in the present instance the pressure is known, and since by the problem, the base and perpendicular of the triangle are given, its area can easily be found.

Thus, the writers on mensuration have shown, that the area of a triangle is equal to half the product of the base drawn into the perpendicular altitude; consequently, if a be put to denote the area of the triangle ABC, we shall have

$$a = \tfrac{1}{2} b l;$$

therefore, by division, the perpendicular depth of the centre of gravity, is

$$d = \frac{\frac{1}{6} b l^2}{\frac{1}{2} b l} = \tfrac{1}{3} l.$$

And in like manner it may be shown, that if the side BC were horizontal, the perpendicular depth of the centre of gravity would be

$$d = \tfrac{1}{3} b;$$

therefore, take Bm and Bn respectively equal to one third of BC and BA, and through the points m and n, draw mG and nG parallel to BA and BC, meeting each other in the point G; then is G the position of the centre of gravity.

The intelligent and attentive reader will readily perceive that the above determination is not legitimate, since it supposes the pressure upon the triangle ABC to be given; now, this pressure depends entirely upon the position of the centre of gravity, and consequently, the problem supposes the position of the centre of gravity of the triangle ADC to be known; the principle, however, will be more distinctly indicated when applied to other figures, where the above determination may be admitted, without infringing on the precepts of scientific propriety.

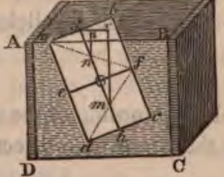
4. OF THE PRESSURE OF INCOMPRESSIBLE FLUIDS ON DIFFERENT SECTIONS OF PARALLELOGRAMS PARALLEL TO THE HORIZON.

PROBLEM V.

43. A rectangular parallelogram is obliquely immersed in an incompressible and non-elastic fluid, in such a manner, that one side is just coincident with the surface :—

> *It is required to compare the pressure on the upper and the lower portions, supposing the parallelogram to be bisected by a line drawn parallel to the surface of the fluid.*

Let AED represent a vessel full of water, or some other non-elastic and incompressible fluid, of which ABEF is the surface, and suppose one side of the vessel to be removed, exhibiting the fluid and the immersed rectangle as represented by ABCD and $abcd$.



Bisect the parallelogram $abcd$ by the straight lines ef and gh respectively parallel to ab and ad; then are $abfe$ and $efcd$, the portions on which the pressures are to be compared, and gh is the line in which the centres of gravity occur.

Draw the diagonals df and fa meeting the straight line gh in the points m and n; then are m and n the centres of gravity of the respective portions into which the parallelogram is divided by the line ef.

Through the point g and in the plane of the fluid surface ABEF, draw gsr at right angles to ab, and the angle mgr will be the inclination or obliquity of the plane; then, through the points m and n, draw the straight lines mr and ns respectively perpendicular to gsr, and rm and sn will be the vertical depths of the centres of gravity below the upper surface of the fluid.

Put $b = ab$, the horizontal breadth of the given parallelogram,
$l = ad$ or bc, the immersed length tending downwards,
$d = rm$, the vertical depth of the centre of gravity of the lower portion $efcd$,
$\delta = sn$, the vertical depth of the centre of gravity of the upper portion $abfe$,
$\phi = mgr$, the inclination of the plane to the surface of the fluid,
$\mathrm{P} =$ the pressure on the whole parallelogram $abcd$,
$p =$ the pressure on the lower portion $efcd$,
and $p' =$ the pressure on the upper portion $abfe$.

Then, because the straight line gh is bisected in $\odot$, and each of the portions $g\odot$ and $h\odot$ respectively bisected in the points n and m; it follows that $gn = \frac{1}{4}$, and $gm = \frac{3}{4}$ of gh; that is

$$gn = \tfrac{1}{4}\,l, \text{ and } gm = \tfrac{3}{4}\,l;$$

consequently, by the principles of Plane Trigonometry, we have

$$sn = \delta = \tfrac{1}{4} l \sin.\phi, \text{ and } rm = d = \tfrac{3}{4} l \sin.\phi;$$

therefore, since the area of each portion of the parallelogram is expressed by $\frac{1}{2}\,bl$, the pressure on each portion is as below, viz.

The pressure perpendicular to the surface $abfe$, is

$$p' = \tfrac{1}{8} b l^2 \sin.\phi,$$

and the pressure perpendicular to the surface $efcd$, is

$$p = \tfrac{3}{8} b l^2 \sin.\phi;$$

consequently, by comparison, the pressures on the upper and lower portions of the parallelogram, are to each other as the numbers 1 and 3; that is

$$p' : p :: 1 : 3.$$

But according to the third problem, the aggregate pressure sustained by the plane, in a direction perpendicular to its surface, is

$$\mathrm{P} = \tfrac{4}{8} b l^2 \sin.\phi;$$

consequently, the pressures on the two portions and on the whole plane, are to one another as the numbers 1, 3 and 4.

In the preceding values of the pressure, it is supposed, that the specific gravity of the fluid in which the plane is immersed, is represented by unity, which is true only in the case of water; therefore, in order to render the formulæ general, we must introduce the symbol for the specific gravity, and then the above equations become,

1. For the upper half of the parallelogram,

$$p' = \tfrac{1}{8}\, b l^2\, s \sin.\, \phi. \qquad (13).$$

2. For the lower half of the parallelogram,

$$p = \tfrac{3}{8}\, b\, l^2 s \sin.\, \phi. \qquad (14).$$

When the plane is perpendicularly immersed in the fluid, or when $\phi = 90°$, then $\sin.\, \phi = 1$, and the equations (13) and (14) become

$$p' = \tfrac{1}{8} b\, l^2 s, \text{ and } p = \tfrac{3}{8} b\, l^2 s.$$

In which equations the co-efficients or constant quantities remain; therefore, the ratio of the pressure is not varied in consequence of a change in the angle of inclination, the variation takes place in the magnitude of the pressures only, and not in the ratio, the magnitude increasing from zero, where the plane is horizontal, to its maximum where the plane is perpendicular.

44. The practical rules for calculating the pressures, as derived by the equations (13) and (14) are as follows.

1. For the pressure on the first, or upper half of the parallelogram.

   Rule. *Multiply the square of the immersed length, by the breadth drawn into the specific gravity of the fluid, and again by the natural sine of the angle of elevation; then, one eighth part of the product will be the pressure sought.* (Eq. 13).

2. For the pressure on the second, or lower half of the parallelogram.

   Rule. *Multiply the square of the immersed length, by the breadth drawn into the specific gravity of the fluid, and again by the natural sine of the angle of elevation; then, three eighths of the product will be the pressure sought.* (Eq. 14).

45. Example 6. A rectangular parallelogram, whose sides are respectively 20 and 30 feet, is immersed in a cistern of water, in such a manner, that its breadth or shorter side is just coincident with the surface; required the pressures on the upper and lower portions of the plane, supposing it to be bisected by a line drawn parallel to the horizon, the inclination of the plane being 59° 38′?

Here, by operating according to the rule, we have

$$p' = 30^2 \times 20 \times \sin.\, 59° \, 38' \times \tfrac{1}{8};$$

but by the Trigonometrical Tables, the natural sine of 59° 38′ is .86281; therefore, we have

$$p' = 30^2 \times 20 \times .86281 \times \tfrac{1}{8} = 1941.3225,$$

and again, by a similar process we have

$$p = 30^2 \times 20 \times .86281 \times \tfrac{3}{8} = 5823.9675.$$

Now, these results are obviously expressed in cubic feet of water, for they are respectively equal to the solidity of a fluid column, whose base is equal to one half the given parallelogram, and whose altitude, in the one case, is expressed by $\frac{1}{4}l \sin. \phi = 7.5 \times .86281$, and in the other by $\frac{3}{4}l \sin. \phi = 22.5 \times .86281$; but the weight of one cubic foot of water is equal to 62½ lbs.; consequently, the pressures expressed in lbs. avoirdupois, are

$$p' = 1941.3225 \times 62.5 = 121332.65625 \text{ lbs.}$$
$$\text{and } p = 5823.9675 \times 62.5 = 363997.96875 \text{ lbs.}$$

When the plane is perpendicular to the surface of the fluid, the pressure is a maximum, and in that case, the respective pressures on the two portions of the parallelogram, are

$$p' = 30^2 \times 20 \times 62.5 \times \tfrac{1}{8} = 140625 \text{ lbs.}$$
$$\text{and } p = 30^2 \times 20 \times 62.5 \times \tfrac{3}{8} = 421875 \text{ lbs.}$$

and the sum of these, is obviously equal to the whole pressure on the plane; hence we get

$$\text{P} = 140625 + 421875 = 562500 \text{ lbs.}$$

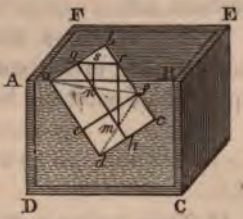
Corol. If the plane, instead of being immersed in the fluid, as we have hitherto supposed it to be, should only be in contact with it, as we may conceive the surface of a vessel to be in contact with the fluid which it contains; then, the pressure will be the same; for the quantity of pressure at any given depth upon a given surface, is always the same, whether the surface pressed be immersed in the fluid or just in contact with it, and whether it be parallel to the horizon, or placed in a position perpendicular or oblique to it.

5. OF RECTANGULAR PARALLELOGRAMS IMMERSED IN NON-ELASTIC FLUIDS, AND DIVIDED INTO TWO PARTS SUCH THAT THE PRESSURES OF THE FLUID UPON THEM SHALL BE EQUAL BETWEEN THEMSELVES.

PROBLEM VI.

46. A rectangular parallelogram is obliquely immersed in an incompressible and non-elastic fluid, in such a manner, that one side is just coincident with the surface:—

> *It is required to divide the parallelogram into two parts by a line drawn parallel to the horizon, so that the pressures on the two parts shall be equal to one another.*



Let AED represent a rectangular vessel filled with water, or some other incompressible and non-elastic fluid, of which ABEF is the surface, and ABCD the fluid as exhibited in the vessel, on the supposition that one of its upright sides is removed.

Let $abcd$ be the immersed parallelogram, having its upper side ab coincident with the surface of the fluid, and its plane tending obliquely downwards in the given angle of inclination. Bisect ab in g, and through g draw the straight line gh parallel to ad or bc, the side of the given immersed rectangle, and let ef parallel to ab or cd, denote the line of division; then, by the problem, the pressure on the rectangle $abfe$, is equal to the pressure on the rectangle $efcd$.

Draw the diagonals df and fa, cutting the bisecting line gh in the points m and n; then are m and n respectively, the places of the centres of gravity of the spaces $efcd$ and $abfe$. Through the point g and in the plane of the fluid surface, draw gr at right angles to ab, and from m and n demit the straight lines mr and ns, respectively perpendicular to the horizontal line gr; then are sn and rm the perpendicular depths of the centres of gravity of the rectangles $abfe$ and $efcd$ on which the pressures are equal.

Put $b = ab$, the horizontal breadth of the proposed rectangular plane,
$l = ad$ or bc, the immersed length of ditto, or that which tends downwards,
$d = rm$, the vertical depth of the centre of gravity of the lower portion $efcd$,
$\delta = sn$, the vertical depth of the centre of gravity of the upper portion $abfe$,
$\phi = mgr$, the inclination of the plane to the surface of the fluid,
$P =$ the pressure on the entire parallelogram,
$p =$ the pressure on each of the portions into which the parallelogram is divided,
$s =$ the specific gravity of the fluid,
and $x = ae$, the immersed length of the upper portion $abfe$.

Then is $ed = l - x$; $gn = \frac{1}{2}x$, and $gm = x + \frac{1}{2}(l - x)$; consequently, by the principles of Plane Trigonometry, we have

$$sn = \delta = \tfrac{1}{2}x \sin.\phi, \text{ and } rm = d = \{x + \tfrac{1}{2}(l - x)\} \sin.\phi,$$

and moreover, by the principles of mensuration, the area of the upper portion is expressed by bx, and that of the lower portion by $b(l - x)$;

consequently, the absolute pressures as referred to the respective portions, are

$$p = \tfrac{1}{2} b x^2 s \sin. \phi, \text{ and } p = b s (l - x) \{ x + \tfrac{1}{2} (l - x) \} \sin. \varphi;$$

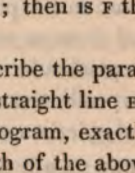
but according to the conditions of the problem, these pressures are equal to one another; hence by comparison, we have

$$\tfrac{1}{2} x^2 = (l - x) \{ x + \tfrac{1}{2} (l - x) \},$$

and this by a little farther reduction, becomes

$$2 x^2 = l^2. \qquad (15).$$

47. The equation in its present form, suggests a very simple geometrical construction; for since l^2 is equal to twice x^2, it is manifest, that l is the diagonal of a square of which the side is x; hence the following process.



Draw the straight line A B of the same length as the side of the given parallelogram, and bisect A B perpendicularly in C by the straight line C D; on A B as a diameter, and about the centre C describe the semicircle A D B, cutting the straight line C D in the point D; join A D, and about the point A as a centre, with the distance A D, describe the arc D E meeting A B in E; then is F the point of division sought.

Upon A B and with the given horizontal breadth, describe the parallelogram A B H G, and through the point E, draw the straight line E F parallel to A G or B H; then will E F divide the parallelogram, exactly after the manner required in the problem. The truth of the above construction is manifest; for by the property of the right angled triangle, we have

$$AD^2 = AC^2 + CD^2;$$

but A C is equal to C D, these being radii of the same circle, hence we get

$$AD^2 = 2\,AC^2;$$

but by the construction, we have

$$AE = AD;$$

consequently, by substitution, it is

$$AE^2 = 2\,AC^2,$$

and doubling both sides of the equation, we get

$$2\,AE^2 = 4\,AC^2;$$

now A C is equal to one half of A B, and it is demonstrated by the writers on geometry, that the square of any quantity is equal to four times the square of its half; consequently, we have

$$4\,AC^2 = AB^2;$$

therefore, by substitution, we obtain

$$2\,AE^2 = AB^2,$$

being the very same expression as that which we obtained by the foregoing analytical process, a coincidence which verifies the preceding construction.

Returning to the equation numbered (15), and extracting the square root of both sides, we obtain

$$x\sqrt{2}=l;$$

and by division, we have

$$x=\tfrac{1}{2}l\sqrt{2}. \qquad (16).$$

48. The practical rule for determining the point of division, as supplied by the above equation, is extremely simple; it may be thus expressed:

> RULE. *Multiply half the length of the immersed side of the parallelogram by the square root of* 2, *or by the constant number* 1.4142, *and the product will express the distance downward from the surface of the fluid.*

49. EXAMPLE 7. A rectangular parallelogram, whose sides are respectively 14 and 28 feet, is immersed in a cistern of water, in such a manner, that its shorter side is just coincident with the surface; through what point in the longer side must a line be drawn parallel to the horizon, so that the pressures on the two parts, into which the parallelogram is divided, may be equal to one another?

Here, by operating according to the rule, we have

$$x=\tfrac{1}{2}(28\times 1.4142)=19.7988 \text{ feet.}$$

50. If the point through which the line of division passes, were estimated in the contrary direction; that is, upwards from the lower extremity of the immersed side of the parallelogram; then, the expression for the place of the point will be very different from that which we have given above, as will become manifest from the following process.

Recurring to the original diagram of Problem 5, and putting $x=ed$, the rest of the notation remaining, we shall have by subtraction,

$$ae=l-x;$$

consequently, sn the depth of the centre of gravity of the rectangle $abfe$, is

$$\delta=\tfrac{1}{2}(l-x)\sin.\phi,$$

and in like manner, it may be shown, that rm, the depth of the centre of gravity of the rectangle $efcd$, is

$$d=(l-\tfrac{1}{2}x)\sin.\phi.$$

Now, according to the writers on mensuration, the area of the rectangle $abfe$ is expressed by $b(l-x)$, and that of the rectangle $efcd$ by bx; consequently, the respective pressures are

$$p = \tfrac{1}{2} b (l - x)^2 s \sin.\phi, \text{ and } p = b x (l - \tfrac{1}{2} x) s \sin.\phi,$$

but by the conditions of the problem, these pressures are equal; hence we get

$$\tfrac{1}{2}(l - x)^2 = x(l - \tfrac{1}{2} x),$$

and this, by reduction, becomes

$$x^2 - 2 l x = - \tfrac{1}{2} l^2;$$

consequently, the root of this equation is

$$x = l - \tfrac{1}{2} l \sqrt{2},$$

and more elegantly, by collecting the terms, it becomes

$$x = l(1 - \tfrac{1}{2} \sqrt{2}). \qquad (17).$$

51. This is manifestly the same result as would arise, by subtracting the value of x in equation (16), from the whole length of the parallelogram; and the rule for performing the operation is simply as follows:

> RULE. *From unity subtract one half the square root of* 2; *then multiply the remainder by the length of the parallelogram, and the product will be the distance of the point required from the lower extremity of the immersed dimension.*

Therefore, by taking the length of the parallelogram, as proposed in the preceding example, we shall have for the distance from its lower extremity, through which the line of division passes,

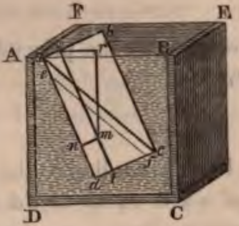
$$x = 28(1 - \tfrac{1}{2} \sqrt{2}) = 8.2012 \text{ feet.}$$

COROL. It is manifest from the equations (16) and (17), that the solution is wholly independent of the breadth of the parallelogram, its inclination to the horizon, and the specific gravity of the fluid; these elements, therefore, might have been omitted in the investigation; but since it became necessary to express the pressure either absolutely or relatively, we thought it better to exhibit the several quantities, of which the measure of the pressure is constituted.

PROBLEM VII.

52. A given rectangular parallelogram is immersed in a fluid, in such a manner, that one side is coincident with the surface, and its plane tending obliquely downwards at a given inclination to the horizon:—

> *It is required to draw a straight line parallel to one of the diagonals, so that the pressures on the parts into which the parallelogram is divided, may be equal to one another.*



Let AED represent a cistern filled with fluid, of which ABEF is the surface, supposed to be perfectly quiescent, and consequently, parallel to the horizon; and let ABCD be a vertical section of the cistern, exhibiting the fluid with the immersed rectangle $abcd$.

Draw the diagonal ac, and in ad take any point e; through the point e thus assumed, draw the straight line ef parallel to ac the diagonal of the parallelogram; then is edf the triangle, on which the pressure is equal to that upon the polygonal figure $eabcf$.

Take dn and dt respectively equal to one third of de and df, and through the points n and t, draw nm and tm respectively parallel to ab and ad, the sides of the parallelogram, and meeting one another in the point m; then, according to problem B, m is the place of the centre of gravity of the triangle edf.

Produce tm directly forward, meeting ab the upper side of the parallelogram perpendicularly in s; then, through the point s, and in the plane of the fluid surface, draw the straight line sr also at right angles to ab, and from m, the centre of gravity of the triangle edf, demit the line mr perpendicularly on sr; then is rm the perpendicular depth of the centre of gravity of the triangle edf, and msr is the angle of inclination of the plane to the horizon.

Put $b = ab$, the horizontal breadth of the given parallelogram,
$l = ad$, the length of the immersed plane tending downwards,
$d = rm$, the perpendicular depth of the centre of gravity of the triangle efd,
$p =$ the whole pressure perpendicular to its surface,
$\phi = msr$, the angle which the immersed plane makes with the horizon,
$s =$ the specific gravity of the fluid,
and $x = ed$, the perpendicular of the triangle edf, of which the base is df.

Then, by reason of the parallel lines ac and ef, the triangles adc and edf are similar to one another, and consequently, by the property of similar triangles, we have

$$ad : dc :: ed : df,$$

which, by restoring the symbols, becomes

$$l : b :: x : df,$$

and from this analogy we have

$$df = \frac{bx}{l};$$

therefore, by the principles of mensuration, the area of the triangle *efd* is

$$\tfrac{1}{2}x \times \frac{bx}{l} = \frac{bx^2}{2l}.$$

Now, according to the construction, *dn* is equal to one third of *ed*, and *an* is equal to *ad* minus *dn*; but *sn* is obviously equal to *an*; hence we have

$$sn = l - \tfrac{1}{3}x,$$

and by the principles of Plane Trigonometry, it is

$$rm = d = (l - \tfrac{1}{3}x) \sin. \phi;$$

consequently, the pressure on the triangle *edf* becomes

$$p = \frac{bx^2 s\,(3l - x) \sin. \phi}{6l},$$

and this, by the conditions of the problem, is equal to half the pressure on the entire parallelogram; therefore, and by equation (7), we have

$$\frac{bx^2 s\,(3l - x) \sin. \phi}{6l} = \frac{bl^2 s \sin. \phi}{4};$$

hence, by expunging the common quantities, we get

$$2x^2(3l - x) = 3l^3,$$

and furthermore, by separating and transposing the terms, it is

$$2x^3 - 6lx^2 = -3l^3,$$

and dividing all the terms by 2, we obtain

$$x^3 - 3lx^2 = -1.5l^3. \qquad (18).$$

It is somewhat remarkable, that the solution of a problem apparently so simple, should require the reduction of a cubic equation; but so it happens, and it may be proper to observe, that in the present instance, it cannot be resolved by means of an equation of a lower degree.

Now, in order to determine the value of x from the above equation, we have only to substitute the numerical value of l as given in the question, and then to resolve the equation by the rules given for that purpose.

53. EXAMPLE 8. Suppose the immersed length of the rectangle, or that tending downwards, to be 20 feet; how far below the surface of the fluid must the point be situated, through which a line drawn parallel to the diagonal, will divide the parallelogram into two parts sustaining equal pressures?

Here the given length is 20; therefore, by substituting 20 and 20^3, respectively for l and l^3 in the above equation, we shall obtain

$$x^3 - 60\,x^2 = -12000.$$

In order, therefore, to take away the term $-60\,x^2$ and prepare the equation for solution, we must put $x = z + 20$, and then by involution, we have

$$x^3 = z^3 + 60\,z^2 + 1200\,z + 8000$$

$$-60\,x^2 = \quad -60\,z^2 - 2400\,z - 24000,$$

from which, by summing the terms, we get

$$x^3 - 60\,x^2 = z^3 - 1200\,z - 16000 = -12000;$$

therefore, by transposition, we obtain

$$z^3 - 1200\,z = 4000.$$

Now, since the equation falls under the irreducible case of cubics, it is manifest, that its solution cannot be effected by Cardan's formula; we must therefore have recourse to some other method, and in the present instance, it will be convenient to adopt the concise and elegant theorem of the *Chevalier de Borda.*

For which purpose, put $a =$ any arc such, that $\text{cosec.}3a = \frac{2m}{3n}\sqrt{\tfrac{1}{3}m}$; then, we shall have

$$z = -2\sqrt{\tfrac{1}{3}m}\ \text{sin.}a, \qquad (19).$$

where m is the co-efficient of the second term, and n the absolute number; consequently, by substitution, we obtain

$$\text{cosec. } 3\,a = \frac{2 \times 1200}{3 \times 4000}\sqrt{400} = 4;$$

therefore, by the Trigonometrical Tables, we have

$$3\,a = 14^\circ\ 28'\ 39'',$$

and by division, we get

$$a = 4^\circ\ 49'\ 33''.$$

But the natural sine of $4^\circ\ 49'\ 33''$ to the radius unity, is 0.08412, and $\tfrac{1}{3}\,m = 400$; consequently, by equation (19), we have

$$z = -40 \times 0.08412 = -3.3648;$$

now, we have seen above, that

$$x = z + 20;$$

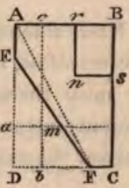
therefore, by substitution, we obtain

$$x = -3.3648 + 20 = 16.6352.$$

Corol. Hence it appears, that if we take 3.3648 feet downwards from the surface of the fluid, or 16.6352 feet upwards from the lower side of the plane, and through the point thus determined in either case, let a straight line be drawn parallel to the diagonal; then shall the rectangle be divided as required in the problem.

6. METHOD OF FINDING THE POSITION OF THE CENTRE OF GRAVITY OF ANY MIXED SPACE OF RECTILINEAR FIGURES IMMERSED IN NON-ELASTIC FLUIDS.

54. Since the pressure on the entire parallelogram, is equivalent to the sum of the pressures on the two parts into which it is divided by the line ef; it follows from thence, that the position of the centre of gravity of the figure $abcfe$ can be determined, as is shown in what follows.



Let ABCFE be the figure, of which the centre of gravity is required to be found, the angles at A, B and C being right angles; join the points A and F by the straight line AF, dividing the figure into two parts, one of which is the triangle AFE and the other the trapezoidal space ABCF.

Now, almost every writer on mechanical science has given the method of finding the centre of gravity of those figures separately, from which that of the compound space may easily be determined; but we are not aware of any method that has been proposed, for the direct discovery of the centre of gravity of the mixed space ABCFE, and that is what we are now about to investigate.

Produce the sides AE and CF till they meet in D; then, because the angles at A, B and C are right angles, the angle at D is also a right angle; from the point D, set off Da and Db, respectively equal to one third of DE and DF, and through the points a and b, draw am and bm parallel to DA and DC, which produce directly forward to d and c; then are cm and dm respectively, the perpendicular depths of the centre of gravity of the triangle EDF, according as the side AB or BC is supposed to be coincident with the surface of the fluid.

Put $b =$ AB, the breadth of the rectangular parallelogram ABCD,
$l =$ AD, the length of ditto,
$\beta =$ DF, the base of the right angled triangle EDF,
$l' =$ DE, the corresponding perpendicular,
$d = cm$, the perpendicular depth of the centre of gravity of the triangle EDF, when the side AB is horizontal,
$\delta = dm$, the perpendicular depth of the centre of gravity m, when the side BC is horizontal,
$p =$ the pressure perpendicular to the surface of the triangle EDF,
$p' =$ the pressure on the irregular figure ABCFE,

$P =$ the pressure on the entire parallelogram ABCD,
$x = rn$, the perpendicular depth of the centre of gravity of the figure ABCFE, when the side AB is horizontal,
and $y = sn$, the perpendicular depth, when the side BC is horizontal.

Then, because the sides AE and CF are given quantities, it follows, that DE and DF are also given, and consequently, Aa or Cm, and Cb or dm are given; therefore, the perpendicular pressure on the triangle EDF can easily be ascertained.

Now, Aa is manifestly equal to the difference between AD and aD, and by the construction aD is equal to one third of DE; therefore, by restoring the analytical representatives, we have

$$\mathrm{c}m = d = l - \tfrac{1}{3} l'.$$

Again Cb is equal to the difference between CD and Db; but Db by the construction, is equal to one third of DF; hence, by restoring the analytical symbols, we shall obtain

$$dm = \delta = b - \tfrac{1}{3}\beta.$$

But, according to the writers on mensuration, the area of the triangle EDF is equal to half the product of the base DF by the perpendicular DE; that is

$$\tfrac{1}{2} l' \times \beta = \tfrac{1}{2} l' \beta;$$

consequently, if we suppose the plane to be perpendicularly immersed in the fluid, while the side AB is coincident with its surface; then, the pressure on the triangle EDF becomes

$$p = \tfrac{1}{6} \beta l' s (3l - l').$$

Now, the pressure on the irregular figure ABCFE, is obviously equal to the difference between the pressures on the entire parallelogram ABCD, and the triangle EDF; but the pressure on the entire parallelogram, according to equation (8), is

$$P = \tfrac{1}{2} b l^2 s;$$

consequently, by subtraction, the pressure on the figure ABCFE, becomes

$$p' = \tfrac{1}{2} b l^2 s - \tfrac{1}{6} \beta l' s (3l - l');$$

but its area is also equal to the difference between that of the parallelogram and triangle; therefore, we obtain $\tfrac{1}{2}(2bl - \beta l')$ for the area of the irregular figure ABCFE; consequently, by division, the perpendicular depth of the centre of gravity below the line AB, becomes

$$x = \frac{3bl^2 s - \beta l' s (3l - l')}{3(2bl - \beta l')};$$

and if we suppose the fluid in which the plane is immersed to be water, the specific gravity of which is unity, we finally obtain

$$x = \frac{3bl^2 - \beta l'(3l - l')}{3(2bl - \beta l')}. \qquad (20).$$

Again, if we suppose the side BC to be horizontal, the area of the triangle remains the same, and the pressure which it sustains in a direction perpendicular to its surface, becomes

$$p = \tfrac{1}{6}\beta l' s(3b - \beta).$$

But the pressure on the whole parallelogram ABCD, on the supposition that the side BC is horizontal, according to what has been proved in Problem 3, is

$$P = \tfrac{1}{2}b^2 l s;$$

consequently, the pressure on the irregular figure ABCFE, becomes

$$p' = \tfrac{1}{2}b^2 l s - \tfrac{1}{6}\beta l' s(3b - \beta).$$

Now, the area of the figure corresponding to the above pressure, is obviously the same as we have previously determined it to be; that is, the difference between the areas of the triangle and the entire parallelogram; consequently, by division, we shall obtain

$$y = \frac{3b^2 l - \beta l'(3b - \beta)}{3(2bl - \beta l')}. \qquad (21).$$

The equations (20) and (21) are manifestly symmetrical; if therefore, we carefully attend to the conditions of the problem, from which they are respectively derived, the position of the centre of gravity of the figure ABCFE can easily be ascertained by resolvng the equations.

55. The practical rules for determining the co-ordinates which fix the position of the centre of gravity, may be expressed in the following manner:

1. When the side AB is horizontal, as indicated by equation (20).

   RULE. *From three times the vertical length of the given rectangular parallelogram, subtract the perpendicular of the triangle, and multiply the remainder by twice its area; then, subtract the product from three times the square of the length of the parallelogram drawn into its breadth, and the remainder will be the dividend.*

   Divide the dividend above determined, by three times the difference between twice the area of the parallelogram, and twice that of the triangle, and the quotient will give the co-ordinate of the line AB.

2. When the side BC is horizontal, as indicated by equation (21).

   RULE. *From three times the vertical breadth of the parallelogram, subtract the base of the triangle, and multiply the* *remainder by twice its area; then, subtract the product from three times the square of the breadth of the parallelogram drawn into its length, and the remainder will be the dividend.*

Divide the dividend above determined, by three times the difference between twice the area of the parallelogram, and twice that of the triangle, and the quotient will give the co-ordinate of the line BC.

56. EXAMPLE 9. The sides of a rectangular parallelogram are respectively 28 and 50 feet, and from one of the lower corners, is separated a right angled triangle, by means of a straight line terminating in the adjacent sides; it is required to determine the position of the centre of gravity of the remaining part, the base and perpendicular of the separated triangle, being respectively equal to 20 and 42 feet?

Here then, by operating as directed in the first rule, we have

$$3 \times 50 - 42 = 150 - 42 = 108,$$

and by the principles of mensuration, twice the area of the triangle, is

$$42 \times 20 = 840 \text{ square feet};$$

therefore, by multiplication, we obtain

$$108 \times 840 = 90720.$$

Again, three times the square of the length of the parallelogram, is

$$3 \times 50^2 = 7500,$$

which being multiplied by its breadth, gives

$$7500 \times 28 = 210000;$$

consequently, by subtraction, the dividend is

$$210000 - 90720 = 119280.$$

Now, twice the area of the parallelogram, is $2 \times 50 \times 28 = 2800$ square feet, and twice the area of the triangle, is $42 \times 20 = 840$ square feet; therefore, by the second clause of the rule, we obtain

$$x = \frac{119280}{3(2800 - 840)} = 20.286 \text{ feet nearly.}$$

Hence it appears, that the co-ordinate of the line AB, according to the proposed data, is very nearly 20.286 feet; and by operating as directed in the second rule, we shall have

$$3 \times 28 - 20 = 84 - 20 = 64,$$

and by the principles of mensuration, twice the area of the triangle, is

$$42 \times 20 = 840 \text{ square feet};$$

therefore, by multiplication, we obtain

$$64 \times 840 = 53760.$$

Again, three times the square of the breadth of the parallelogram, is

$$3 \times 28^2 = 2352,$$

which being multiplied by its length, gives

$$2352 \times 50 = 117600;$$

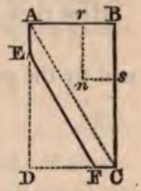
consequently, by subtraction, the dividend becomes

$$117600 - 53760 = 63840.$$

Now, the second clause of the second rule, being the same as the second clause of the first rule, it follows, that the divisor must here be the same, as we have found it to be in the preceding case; consequently, by division, we obtain

$$y = \frac{63840}{5880} = 10.857 \text{ feet};$$

therefore, from the numerical values of the co-ordinates as we have just determined them, the position of the centre of gravity of the proposed figure can easily be found, in the following manner.



57. Let ABCD represent the rectangular parallelogram, of which the side AB is 28 feet, and the side BC 50 feet; and let EDC be the right angled triangle, whose perpendicular ED is 42 feet, and its base DF 20 feet, all taken from the same scale of equal parts.

From the angle B, and on the sides BC and BA, set off B*s* and B*r* respectively equal to 20.286 and 10.857 feet; then, through the points *s* and *r*, draw the lines *sn* and *rn*, respectively parallel to AB and BC, and the point *n* is the centre of gravity of the figure ABCFE, which remains after the right angled triangle EDF is separated from the parallelogram ABCD.

If the line of division, or hypothenuse of the triangle EF, were parallel to AC the diagonal of the parallelogram, as is distinctly specified in the foregoing problem, the solution would become much more simple; for then, in order to determine the position of the centre of gravity, it is only necessary to reduce one of the equations, and it is altogether a matter of indifference which of them it is, provided that the conditions of the equation be strictly attended to.

Supposing ED the perpendicular of the triangle, to remain as above; then the base, when the hypothenuse is parallel to the diagonal of the rectangle, will be found by the following analogy, viz.

$$50 : 28 :: 42 : 23.52.$$

Then, by calculating according to rule first, or equation (20), our dividend and divisor are 103313.28 and 5436.48 respectively; consequently, we get

$$x = \frac{103313.28}{5436.48} = 19 \text{ feet nearly};$$

therefore, by analogy, we obtain

$$50 : 19 :: 28 : y = 10.64 \text{ feet.}$$

Here, the whole process of calculating the second co-ordinate, is replaced by the simple analogy above exhibited.

The example now before us, affords a striking instance of the advantages to be derived from this mode of considering the centre of gravity; in the case of the triangle illustrated under Problem (B), its immediate utility was not so conspicuously displayed; but we are convinced, that in figures of more difficult and complicated forms, its usefulness will become still more evident.

In the investigation of the formulæ, we have thought it necessary to consider the pressure on the surface whose centre of gravity is sought; but in the actual application of the resulting equations, the consideration of pressure does not enter; for it is manifest, that besides the dimensions of the figure and constant numbers, no other element is found in the equations, and consequently, the reduction depends upon them alone.

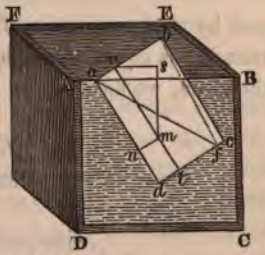
7. OF EQUAL FLUID PRESSURES ON THE SECTIONS OF A RECTANGULAR PARALLELOGRAM AND THE PERPENDICULAR DEPTHS OF THE CENTRE OF GRAVITY.

PROBLEM VIII.

58. A given rectangular parallelogram is immersed in an incompressible and non-elastic fluid, in such a manner, that one of its sides is coincident with the surface, and its plane tending downwards at a given inclination to the horizon:—

> *It is required to draw a straight line from one of the upper angles to the lower side, so that the pressures on the two parts into which the parallelogram is divided, may be equal to one another.*

Let AED represent a rectangular cistern filled with water, or some other incompressible and non-elastic fluid, of which ABEF is the horizontal surface, and suppose one of the upright sides, as ABCD to be removed, exhibiting the fluid together with the immersed rectangle *abcd*.



In *dc* the lower side of the immersed parallelogram, take any point *f*, and draw *af* to represent the line of division; then the triangle *adf*, and the trapezoid *abcf*, are the figures into which the parallelogram is divided, and on which the pressures are equal.

From the angle *d*, set off *dn* and *dt* respectively equal to one third of *da* and *df*, and through the points *n* and *t*, draw *nm* and *tm* parallel to *df* and *da* the sides of the triangle, and meeting each other in the point *m*; then, according to what has been demonstrated in Problem (B), *m* is the centre of gravity of the triangle *adf*.

Produce *tm* directly forward, meeting *ab* at right angles in the point *r*, and through the point *r* and in the plane of the fluid surface, draw *rs* also at right angles to *ab*, and demit *ms* meeting *rs* perpendicularly in *s*; then is *mrs* the inclination of the plane to the horizon, and *sm* the perpendicular depth of the centre of gravity of the triangle *adf* below the upper surface of the fluid.

Put $b = ab$, the horizontal breadth of the parallelogram *abcd*,
$l = ad$, the immersed length tending downwards,
$p =$ the pressure perpendicular to the surface of the triangle *adf*.
$\mathrm{P} =$ the pressure on the entire parallelogram *abcd*,
$\phi = mrs$, the angle which the immersed plane makes with the horizon,
$d = sm$, the perpendicular depth of the centre of gravity of the triangle *adf*,
$s =$ the specific gravity of the fluid,
and $x = df$, the distance between *d* and the point through which the line of division passes.

Then, according to the principles of Plane Trigonometry, the perpendicular depth of the centre of gravity of the triangle *adf*, becomes

$$d = \tfrac{2}{3} l \sin.\phi;$$

consequently, the pressure on its surface, is

$$p = \tfrac{1}{3} l^2 x s \sin.\phi; \ \{ \textit{see equation } (10) \}.$$

But according to equation (7) under the 3rd problem, (art. 33), the pressure on the entire rectangle, is

$$\mathrm{P} = \tfrac{1}{2} b l^2 s \sin.\varphi;$$

and by the conditions of the present problem, the pressure on the triangle, is equal to one half the pressure on the entire parallelogram; therefore, we have

$$p = \tfrac{1}{2}\mathrm{P}; \text{ that is}$$

$$\tfrac{1}{3} l^2 x s \sin.\phi = \tfrac{1}{4} b l^2 s \sin.\phi,$$

from which, by expunging the common terms, we get

$$4x = 3b;$$

consequently, by division, we obtain

$$x = \frac{3b}{4}. \qquad (22).$$

59. This equation is too simple in its arrangement to require any formal directions for its resolution; nevertheless, the following rule may be useful to many of our readers.

> RULE. *Take three fourths of that side of the given rectangular parallelogram, in which the line of division terminates, and the point thus discovered, is that through which the line of division passes.*

60. EXAMPLE 10. A rectangular parallelogram, whose sides are respectively equal to 24 and 42 feet, is immersed in a cistern full of water, in such a manner, that its shorter side is coincident with the surface of the fluid, and its plane inclined to the horizon in an angle of 52 degrees; it is required to determine a point in its lower side, to which, if a straight line be drawn from one of the upper angles, the parallelogram shall be divided into two parts sustaining equal pressures; and moreover, if a straight line be drawn from the same point in the lower side, to the other upper angle, it is required to assign the pressure on the triangle thus cut off?

Here, by operating according to the rule, the point of division is

$$x = \tfrac{3}{4} \times 24 = 18 \text{ feet.}$$

In the next place, to determine the pressure sustained by the triangle bcf, cut off from the parallelogram $abcd$, by means of the line fb drawn from the point f to the angle at b, we have according to equation (12), (Problem 4),

$$p' = \tfrac{1}{3}(b - x)\, l^2 s \sin.\phi,$$

where $(b - x)$ in this equation, takes place of b in the one referred to.

The natural sine of 52 degrees according to the Trigonometrical Tables, is .78801; hence, by substituting the respective data in the above equation, we shall have

$$p' = \tfrac{1}{3}(24 - 18) \times 42^2 \times .78801 = 2746.09928 \text{ cub. ft. of}$$

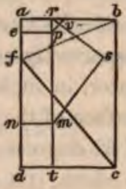
water; consequently, the pressure expressed in lbs. avoirdupois, is

$$p' = 2746.09928 \times 62.5 = 181631.205 \text{ lbs.}$$

This seems to be an immense pressure, on a triangle whose surface is only 126 square feet; it is however but one sixth part of the pressure on the entire parallelogram; this is manifest, for the pressure on the triangle adf, is three times the pressure on the triangle bcf, since the base df is equal to three times the base cf, and the altitudes of the triangles, as well as the perpendicular depths of the centres of gravity, are the same; but the pressure on the parallelogram $abcd$, according to the problem, is double the pressure on the triangle *adf*; hence we have

$$P = 2p = 6p' = 1089787.23 \text{ lbs.}$$

61. If the line of division were drawn from one of the lower angles to a point in the immersed length, after the manner represented in the annexed diagram; then, the equation (22), would assume a different form, as will become manifest from the following investigation.



From the angle *d* on *da* and *dc* the sides of the parallelogram, set off *dn* and *dt*, respectively equal to one third of *df* and *dc*, and through the points *n* and *t* thus found, draw the straight lines *nm* and *tm* parallel to *dc* and *df*, the base and perpendicular of the triangle *fdc*, and meeting one another in *m*, the place of its centre of gravity.

Produce *tm* directly forward, meeting *ab*, the horizontal side of the given parallelogram perpendicularly in the point *r*; at the point *r* in the straight line *mr*, make the angle *mrs* equal to the angle of the plane's inclination, and draw *ms* perpendicularly to *rs*; then is *sm* the perpendicular depth of the centre of gravity of the triangle *fdc*.

Let therefore, the notation of the preceding case be retained, and put $x = df$; then we have

$an = rm = l' - \frac{1}{3}x$, and consequently $sm = d = (l - \frac{1}{3}x)\sin.\phi$;

but the area of the triangle *fdc* is expressed by $\frac{1}{2}bx$; therefore, the pressure perpendicular to its surface, is

$$p = \tfrac{1}{6}bxs(3l - x)\sin.\phi;$$

now, according to the conditions of the problem, the pressure on the separated triangle is equal to half the pressure on the entire parallelogram; consequently, we obtain

$$\tfrac{1}{6}bxs(3l - x)\sin.\phi = \tfrac{1}{4}bl^2 s\sin.\phi,$$

and this, by expunging the common quantities, becomes

$$2x(3l - x) = 3l^2,$$

or dividing by 2 we get

$$x(3l - x) = 1.5l^2,$$

and from this, by separating and transposing the terms, we have

$$x^2 - 3lx = -1.5l^2. \qquad (23).$$

If the equations (18) and (23) be compared with one another, it will readily appear, that they are precisely similar in form, but different in degree; the former being an incomplete cubic, wanting the first power of the unknown quantity, and the latter an adfected quadratic, having all its terms. Indeed, the diagrams from which the two equations are derived, as well as the specified conditions of the problems, are nearly similar, the difference consisting simply in the position of the dividing line, it being parallel to the diagonal of the parallelogram in the one case, and oblique to it in the other.

62. Let the quantity $2\frac{1}{4}l^2$ be added to both sides of the preceding equation, and we shall obtain

$$x^2 - 3\,l\,x + 2\tfrac{1}{4}l^2 = \tfrac{3}{4}l^2,$$

from which, by extracting the square root, we get

$$x - 1\tfrac{1}{2}l = +\tfrac{1}{2}l\sqrt{3};$$

therefore, by transposition, we have

$$x = \tfrac{1}{2}l\,(3 - \sqrt{3}). \qquad (24).$$

The practical rule by which the point of division is to be determined, may be expressed as follows :

> RULE. *Multiply the difference between* 3 *and the square root of* 3, *by half the length of that side of the parallelogram in which the line of division terminates, and the product will be the distance of the required point from the lower extremity of the given length.*

63. EXAMPLE 11. Let the numerical data remain precisely as in the preceding case; from what point in the length of the parallelogram, must a straight line be drawn to the opposite lower angle, so that the parallelogram may be divided into two parts sustaining equal pressures; and moreover, if a straight line be drawn from the same point, to the opposite upper angle, what will be the pressure on the triangle thus cut off?

Here, by proceeding as directed in the above rule, we have

$$x = 21\,(3 - \sqrt{3}) = 26.628 \text{ feet.}$$

In order to find the pressure on the triangle abf cut off by the line bf, we have $af = l - x$, and $ae = rp = \frac{1}{3}(l - x)$; consequently, $vp = \frac{1}{3}(l - x)\sin.\phi$, where it must be observed, that ep and vp are respectively parallel to ab and sm.

Now, the pressure perpendicular to the surface of the triangle abf, is found by multiplying its area into vp, the perpendicular depth of its centre of gravity; hence, we have

$$p' = \tfrac{1}{6}b\,s\,(l - x)^2 \sin.\phi;$$

but the value of x, according to equation (24), is

$$x = .634\,l;$$

consequently, by substitution, we have

$p' = \frac{1}{6}b\,l^2\,s\,(1 - .634)^2 \sin.\phi$, from which, by substituting the several numerical values, we obtain

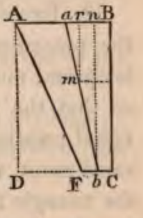
$$p' = 4 \times 42^2 \times 62.5 \times .366^2 \times .78801 = 46551.35 \text{ lbs.}$$

8. PERPENDICULAR DEPTH OF THE CENTRE OF GRAVITY OF A PARALLELOGRAM DIVIDED INTO TWO PARTS SUSTAINING EQUAL PRESSURES.

64. With respect to the centre of gravity of the figure *abcf*, which remains after the triangle *adf*, or *fdc* has been separated from the parallelogram, it is in this particular instance very easily determined; for, since the area of trapezoids, whose parallel sides and perpendicular breadths are equal each to each, are also equal; it follows, that the centre of gravity must occur in the straight line which bisects the parallel sides; it is therefore, only necessary to investigate the theorem for calculating one of the co-ordinates, the other being determinable from the circumstance just stated.

Let ABCF be the trapezoid, having the angles at B and C respectively right angles, and of which the position of the centre of gravity is required.

Produce the side CF directly forward to any convenient length at pleasure, and through the point A, draw the straight line AD parallel to BC, the longer side of the trapezoidal figure, and meeting CF produced perpendicularly in the point D.



Then, the pressure upon the trapezoid ABCF, is manifestly equal to the difference between the pressure on the parallelogram ABCD, and that upon the triangle ADF, and its area, is also equal to the difference between their areas. Bisect the parallel sides AB and CF in the points *a* and *b*, and join *ab*; then, according to what has been demonstrated by the writers on mechanics, the centre of gravity of the trapezoid ABCF occurs in the straight line *ab*.

Suppose it to occur at *m*, and through the point *m* draw *mr* and *ms* respectively parallel to BC and BA, meeting AB and BC perpendicularly in the points *a* and *b*; then are *rm* and *sm* the co-ordinates, whose intersection determines the position of the point *m*.

Put $b = AB$, the breadth of the parallelogram ABCD,
$l = AD$, or BC, its corresponding length,
$\delta = mr$ the depth of the point *m* as referred to the line AB considered to be horizontal,
$\delta' = ms$, the depth of the point *m* as referred to the line BC under similar circumstances,
and $\beta = DF$, the base of the triangle ADF.

Then, by conceiving the plane to be immersed perpendicularly in a fluid whose specific gravity is expressed by unity, the pressure upon the entire surface ABCD, according to equation (8) under the third problem, becomes

$$P = \tfrac{1}{2} b\, l^2;$$

and moreover, by equation (12) under the fourth problem, the general expression for the pressure on the triangle ADF, is

$$p = \tfrac{1}{3} \beta\, l^2 s \sin.\phi\,;$$

but according to the particular case now under consideration, the above expression becomes

$$p = \tfrac{1}{3} \beta\, l^2,$$

the terms s and $\sin.\phi$, being each equal to unity, they disappear in the equation.

Now, according to what we have stated above, the pressure on the trapezoid ABCF, is equal to the difference between the pressure on the entire parallelogram ABCD, and that on the triangle ADF; that is

$$p' = P - p = \tfrac{1}{2} b\, l^2 - \tfrac{1}{3} \beta\, l^2;$$

or, by reducing the fractions to a common denominator and collecting the terms, we obtain

$$p' = \tfrac{1}{6} l^2 (3\,b - 2\,\beta).$$

By the principles of mensuration, the area of the trapezoid ABCF, is equal to the product that arises, when half the sum of the parallel sides AB and CF, is multiplied by BC the perpendicular distance between them; that is,

$$\text{BC} \times \tfrac{1}{2} (\text{AB} + \text{CF}) = \tfrac{1}{2} l\, (2b - \beta),$$

and the perpendicular depth of the centre of gravity, is equal to the pressure on the surface, divided by the area of the figure; consequently, we obtain

$$\delta = \frac{l\,(3\,b - 2\,\beta)}{3\,(2\,b - \beta)}\,. \qquad (25).$$

The form of this equation is extremely simple, but it may be arrived at independently of the preceding investigation, by having recourse to equation (20) under Problem 6; for according to the conditions of the question, the line of division AF originates at the angle A, and consequently, the perpendicular of the triangle and the length of the parallelogram are equal; therefore, by putting l instead of l' in equation (20), the above expression immediately obtains.

Now, by taking the length and breadth of the parallelogram, as given in the preceding example, and the base of the triangle as computed by equation (22), we shall obtain,

$$\delta = \frac{42\,(3 \times 24 - 2 \times 18)}{3\,(2 \times 24 - 18)} = 16.8 \text{ feet.}$$

65. Having thus determined the magnitude of the co-ordinate Bs or rm from the equation (25), the magnitude of the corresponding

co-ordinate Br or sm, can very easily be found; for through the point b the bisection of FC, draw bn parallel to BC and meeting AB perpendicularly in n; then, the triangles ban and mar are similar to one another, and the sides bn, mr and an, are given to find ar, and from thence the rectangular co-ordinate Br or sm; consequently, we have

$$ba : na :: mr : ra;$$

therefore, by subtraction, we get

$$\text{B}r \text{ or } sm = a\text{B} - ra.$$

Now, bC $=$ Bn, is obviously equal to half the difference between DC, the breadth of the parallelogram, and DF, the base of the triangle ADF; therefore, we have

$$\text{B}n = \tfrac{1}{2}(b - \beta);$$

but $an = a\text{B} - \text{B}n$; that is, $an = \frac{1}{2}\beta$, and

$$l : \tfrac{1}{2}\beta :: \frac{l(3b - 2\beta)}{3(2b - \beta)} : ra;$$

therefore, by reducing the analogy, we get

$$ra = \frac{\beta(3b - 2\beta)}{6(2b - \beta)};$$

hence, by subtraction, we obtain

$$\delta' = \frac{3b(b - \beta) + \beta^2}{3(2b - \beta)}. \qquad (26).$$

After the same manner that equation (25) is deducible from equation (20), by putting $l' = l$; so also, is equation (26) deducible from equation (21), by means of the same equality; we might therefore have dispensed with the preceding investigation, and derived the expression from principles already established; we however preferred obtaining it as above, for the purpose of exhibiting that agreeable variety which gives additional embellishment to scientific investigations. The method of establishing the formulæ, on the supposition that the side BC is horizontal, is sufficiently obvious from what has been done in the sixth problem preceding, and therefore, it need not be repeated here.

Corol. By substituting the numerical values of b and β, as given in the preceding example, we shall have from equation (26)

$$\delta' = \frac{3 \times 24(24 - 18) + 18^2}{3(2 \times 24 - 18)} = 8.4 \text{ feet.}$$

Therefore, from the point B, set off Bs and Br respectively equal to 16.8 and 8.4 feet; and through the points s and r, draw sm and rm parallel to AB and BC, the perpendicular sides of the given trapezoid, and meeting one another in the point m; then is m the required place of the centre of gravity.

66. In computing numerically the values of the rectangular co-ordinates mr and ms, we have supposed, that DF the base of the applied triangle, is determinable by the application of equation (20); this supposition however is perfectly unnecessary, for the base of the triangle is always equal to the difference between the parallel sides of the given trapezoid; and moreover, the equation (20), applies only to the particular case for which it has been deduced, viz. when the pressure on the applied triangle and that on the trapezoid to which it is applied are equal to one another.

9. WHEN THE PARALLELOGRAM IS SO DIVIDED, THAT THE PRESSURES ON THE TWO PARTS ARE TO ONE ANOTHER IN ANY RATIO WHATEVER.

67. In the sixth, seventh and eighth problems preceding, we have supposed the given rectangular parallelogram to be divided into two parts, such, that the pressures upon them shall be equal between themselves, and the investigation has accordingly been limited to that particular case; but in order to render the solution general, we shall consider the division to be so effected, that the pressures on the two parts may be to one another in any ratio whatever, such as that of m to n.

For which purpose then, by referring to the fifth problem, where the given parallelogram is divided horizontally, we find, that the pressure on the upper portion is expressed by $\frac{1}{2}bx^2s\sin.\phi$, and that on the lower portion, by $\{\frac{1}{2}(l-x)^2+x(l-x)\}\,bs\sin.\phi$; but these expressions in their present state are equal to one another, and they are now required to be reduced in the ratio of m to n; consequently, we have

$$\tfrac{1}{2}x^2 : \{\tfrac{1}{2}(l-x)^2+x(l-x)\} :: m : n,$$

and this, by expanding the second term, becomes

$$x^2 : l^2-x^2 :: m : n;$$

or by equating the products of the extremes and means, we obtain

$$nx^2=ml^2-mx^2;$$

therefore, by transposition, we get

$$(m+n)\,x^2=ml^2,$$

and finally, by division and evolution, we have

$$x=l\sqrt{\frac{m}{m+n}}. \qquad (27).$$

68. Again, in the case of the sixth problem, where the given parallelogram is divided by a line drawn parallel to the diagonal; we find, that the pressure on the triangle cut off by the line of division, is expressed by $b x^2 s (3l - x) \sin.\phi \div 6l$, and consequently, by subtraction, that on the remaining portion is expressed by $bs \sin.\phi \{ 3l^3 - x^2 (3l - x) \} x \div 6l$; now, these expressions, by the conditions of the problem, are equal to one another; but in the present case, they are to be reduced in the ratio of m to n; for which purpose we have

$$x^2 (3l - x) : 3l^3 - x^2 (3l - x) :: m : n;$$

therefore, by equating the products of the extreme and mean terms, we get

$$n x^2 (3l - x) = 3ml^3 - m x^2 (3l - x);$$

and from this, by transposition, we shall obtain

$$(m + n)(3lx^2 - x^3) = 3ml^3;$$

therefore, by dividing and transposing the terms, we have

$$x^3 - 3lx^2 = -\frac{3ml^3}{m+n}. \qquad (28).$$

Now, in order to reduce the above equation, there must be substituted the numbers which express the given ratio, together with the length of the parallelogram, and then; the value of x will be obtained by any of the rules for resolving cubic equations.

69. In like manner as above, by referring to the eighth problem, where the given parallelogram is divided by a line drawn from one of the upper angles, and terminating in the lower side; we find, that the pressure on the triangle cut off by the line of division, is expressed by $\frac{1}{3} l^2 x s \sin.\phi$, and consequently, by subtraction, the pressure on the remaining portion is expressed by $\frac{1}{6} l^2 s \sin.\phi (3b - 2x)$; and these expressions, according to the conditions of the problem, are equal to one another; but in the present instance, they are to be reduced in the ratio of m to n; hence, we have

$$\tfrac{1}{3} x : \tfrac{1}{6} (3b - 2x) :: m : n;$$

consequently, by equating the products of the extreme and mean terms, we get

$$2nx = 3bm - 2mx,$$

from which, by transposition, we obtain

$$2(m + n) x = 3bm,$$

and finally, by division, we have

$$x = \frac{3bm}{2(m+n)}. \qquad (29).$$

Hence then, the equations (27,) (28,) and (29,) express generally the relation between the parts of division, which in the several problems is restricted to a ratio of equality; and it is presumed, that by paying a due attention to the examples that have been proposed and illustrated, the diligent reader will find no difficulty in resolving any example that may present itself under one or other of the general forms above investigated.

In all the above cases, we have supposed the breadth, or that side of the parallelogram which is denoted by b to be horizontal, and coincident with the surface of the fluid; but it is manifest, that equations of the same form would be obtained from the other side, having b in place of l, and l in place of b.

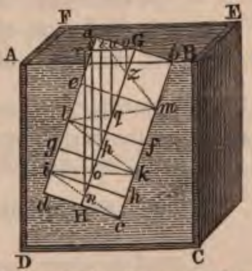
10. OF RECTANGULAR PARALLELOGRAMS DIVIDED INTO SECTIONS SUSTAINING EQUAL PRESSURES; WITH THE METHOD OF DETERMINING A LIMIT TO THE NECESSARY THICKNESS OF FLOOD-GATES, AND OTHER CONSTRUCTIONS OF A SIMILAR NATURE.

PROBLEM IX.

70. A given rectangular parallelogram, is immersed in an incompressible and non-elastic fluid, in such a manner, that one of its sides is coincident with the surface, and its plane inclined at a given angle to the horizon:—

> *It is required to divide the rectangle by lines drawn parallel to the horizon, into any number of parts, such, that the pressures on the several parts of division shall be equal to one another.*

Let AED represent a rectangular cistern filled with water, or some other transparent and incompressible fluid in a state of rest; one side of the vessel being removed, for the purpose of exhibiting the fluid and the immersed parallelogram, together with the several subordinate lines on which the investigation depends.



Suppose e, l, g and i to be the several points of division, and through these points draw the lines em, lf, gk and ih, respectively parallel to ab or dc, the horizontal sides of the figure. Bisect the sides ab and dc in the points G and H; join GH, and draw the zigzag diagonals am, ml, lk, ki and ic, cutting the bisecting line

GH in the points z, q, p, o and n, which points are the respective centres of gravity of the several parts into which the given parallelogram is divided, and on which the pressures are supposed to be equal among themselves.

Through the point G, and in the plane of the horizon, draw the straight line Gr at right angles to ab, making the angle rGH equal to the given angle of the plane's inclination, and from the points n, o, p, q and z, let fall the perpendiculars zv, qu, pt, os and nr, meeting the line Gr respectively in the points v, u, t, s and r; then are the lines zv, qu, pt, os and nr, the perpendicular depths of the centres of gravity of the several portions into which the proposed rectangle is divided, the points of division being estimated from the surface downwards.

Put $b = ab$ or dc, the horizontal breadth of the given parallelogram $abcd$,
$l = ad$ or bc, the entire immersed length, or that tending downwards,
$\phi = rGH$, the given angle of inclination,
$d = vz$, the vertical depth of the centre of gravity of the part $abme$,
$d' = uq$, the vertical depth of ———————— $emfl$,
$d'' = tp$, the vertical depth of ———————— $lfkg$,
$\delta = so$, the vertical depth of ———————— $gkhi$,
$\delta' = rn$, the vertical depth of ———————— $ihcd$;
$n =$ the number of parts into which the parallelogram is divided,
$P =$ the entire pressure on the parallelogram $abcd$,
$p =$ the pressure, common to each of the parts into which the given parallelogram is divided,
$v = ae$, the required length of the upper portion $abme$,
$w = el$, the length of the second portion $emfl$,
$x = lg$, the length of the third portion $lfkg$,
$y = gi$, the length of the fourth portion $gkhi$,
$z = id$, the length of the fifth portion $ihcd$.

Then, according to equation (7) under the third problem, the entire pressure on the parallelogram $abcd$, is

$$P = \tfrac{1}{2} b l^2 s \sin.\phi;$$

therefore, the pressure on each part of the divided figure, is

$$p = \frac{b l^2 s \sin.\phi}{2n}.$$

But because the value of ϕ, the angle of inclination, and s the specific gravity of the fluid, are the same for all the parts; those quantities may be omitted in the equation, and then the element of comparison, or the n^{th} part of the total pressure becomes

$$p=\frac{b\,l^2}{2\,n} \qquad (30).$$

Now, according to the principles of Plane Trigonometry, the lines vz, uq, tp, so and rn, are respectively as below, viz.

$$d=\mathrm{G}z\sin.\phi;\ d'=\mathrm{G}q\sin.\phi;\ d''=\mathrm{G}p\sin.\phi;$$
$$\delta=\mathrm{G}o\sin.\phi,\text{ and }\delta'=\mathrm{G}n\sin.\phi;$$

but the lines $\mathrm{G}z$, $\mathrm{G}q$, $\mathrm{G}p$, $\mathrm{G}o$ and $\mathrm{G}n$, when expressed in terms of the respective lengths, are as follows, viz.

$$\mathrm{G}z=\tfrac{1}{2}v;\ \mathrm{G}q=v+\tfrac{1}{2}w;\ \mathrm{G}p=v+w+\tfrac{1}{2}x;$$
$$\mathrm{G}o=v+w+x+\tfrac{1}{2}y,\text{ and }\mathrm{G}n=v+w+x+y+\tfrac{1}{2}z;$$

therefore, by substitution, the above values of the vertical depths of the respective centres of gravity, become

$$d=\tfrac{1}{2}v\sin.\phi;\ d'=(v+\tfrac{1}{2}w)\sin.\phi;\ d''=(v+w+\tfrac{1}{2}x)\sin.\phi;$$
$$\delta=(v+w+x+\tfrac{1}{2}y)\sin.\phi,\text{ and }\delta'=(v+w+x+y+\tfrac{1}{2}z)\sin.\phi.$$

Consequently, by throwing out the common factor $\sin.\phi$ and neglecting the specific gravity of the fluid, the value of p, or the pressure sustained by each of the parts, may be expressed as follows, viz.

The pressure on the part $abme$, is $p=\frac{1}{2}bv^2$, (1).
———————— $emfl$, is $p=bw\,(v+\frac{1}{2}w)$, (2).
———————— $lfkg$, is $p=bx\,(v+w+\frac{1}{2}x)$, (3).
———————— $gkhi$, is $p=by\,(v+w+x+\frac{1}{2}y)$, (4).
———————— $ihcd$, is $p=bz\,(v+w+x+y+\frac{1}{2}z)$.(5).

Now, according to the conditions of the problem, all these expressions for the value of p, are equal to one another, and each of them is equal to the element of comparison, as given in the equation (30); hence, from the first of the above equations, or values of p, we have

$$\frac{b\,v^2}{2}=\frac{b\,l^2}{2\,n},$$

or by expunging the common factor, $\frac{1}{2}b$, we obtain

$$v^2=\frac{l^2}{n};$$

therefore, by extracting the square root, we have

$$v=\frac{l}{n}\sqrt{n}\cdot$$

Again, by comparing equation (30), with the second of the foregoing expressions for the value of p, we shall have

$$b\,w\,(v+\tfrac{1}{2}w)=\frac{b\,l^2}{2n}\cdot$$

and substituting the value of v in terms of l and n, we obtain

$$n\,w^2+2l\sqrt{n}\,.\,w=l^2,$$

from which, by dividing by n, we get

$$w^2+\frac{2l\sqrt{n}}{n}\,.\,w=\frac{l^2}{n};$$

therefore, if this be reduced by the rule which applies to the resolution of adfected quadratic equations, we shall obtain

$$w=\frac{l}{n}(\sqrt{2n}-\sqrt{n}).$$

Proceed as above, by comparing the equation (30), with the third of the preceding expressions for the value of p, and we shall have

$$b\,x\,(v+w+\tfrac{1}{2}x)=\frac{b\,l^2}{2n}\cdot$$

and if the above values of v and w, as expressed in terms of l and n, be substituted instead of them in this equation, it will become

$$n\,x^2+2l\sqrt{2n}\,.\,x=l^2,$$

and dividing by n, we get

$$x^2+\frac{2l\sqrt{2n}}{n}\,.\,x=\frac{l^2}{n};$$

therefore, by completing the square, evolving and transposing, we obtain

$$x=\frac{l}{n}\{\sqrt{3n}-\sqrt{2n}\}.$$

By pursuing a similar mode of comparison, and reasoning in the same manner, with respect to the fourth value of p foregoing, we shall have

$$b\,y\,(v+w+x+\tfrac{1}{2}y)=\frac{b\,l^2}{2n};$$

let the values of v, w and x, as determined above, be respectively substituted in this equation, and it becomes

$$y^2+\frac{2l\sqrt{3n}}{n}\cdot y=\frac{l^2}{n};$$

complete the square, and we obtain

$$y^2+\frac{2l\sqrt{3n}}{n}+\frac{3l^2n}{n^2}=\frac{4l^2}{n^2},$$

and from this, by evolution and transposition, we get

$$y = \frac{l}{n}\{\sqrt{4n} - \sqrt{3n}\}.$$

Pursuing still the same mode of induction for the fifth value of p, and substituting the respective values of v, w, x and y, as we have determined them above in terms of l and n; then we shall have

$$z = \frac{l}{n}\{\sqrt{5n} - \sqrt{4n}\}.$$

And in like manner we may proceed for any number of divisions at pleasure; but what we have now done is sufficient to exhibit the law of induction.

The formulæ which we have investigated, for determining the several sections of the given parallelogram, may now be advantageously collected into one place; for it is manifest, that by exhibiting them in juxta-position, the law of their formation is more easily detected, and the difference which obtains between the co-efficients of the successive terms becomes at once assignable.

The several equations therefore, when arranged according to the order of the corresponding sections, will stand as under, viz.

$$\left.\begin{array}{ll} 1. & v = \frac{l}{n}(\sqrt{n}), \\ 2. & w = \frac{l}{n}(\sqrt{2n} - \sqrt{n}), \\ 3. & x = \frac{l}{n}(\sqrt{3n} - \sqrt{2n}), \\ 4. & y = \frac{l}{n}(\sqrt{4n} - \sqrt{3n}), \\ 5. & z = \frac{l}{n}(\sqrt{5n} - \sqrt{4n}), \\ & \&c. \&c. = \&c.\ \&c. \end{array}\right. \qquad (31).$$

71. The practical rule for determining the points of section, in reference to their respective distances from the upper extremity of the parallelogram, may be expressed in words, as follows, viz.

Rule. *Multiply the number of parts into which the parallelogram is proposed to be divided, by the number that indicates the place of any particular section; then, multiply the square* *root of the product by the length of the parallelogram, and divide by the whole number of sections, and the quotient will express the distance of any particular point from the upper extremity of the divided length.*

If the length of any particular section, or the distance between any two contiguous points should be required, which is the condition expressed by each of the above equations; then,

Calculate for each of the points according to the preceding rule, and the difference of the results will give the length of the required section.

72. EXAMPLE 12. A rectangular parallelogram whose length is 25 feet, is perpendicularly immersed in a fluid, in such a manner, that its breadth or upper side is just in contact with the surface; now, if it be proposed to divide the parallelogram by lines drawn parallel to the horizon, into five parts sustaining equal pressures; it is required to determine the distance of each point of section from the surface of the fluid, and the respective distances between the several points?

Here then, we have given $l = 25$ feet, and $n = 5$ an abstract number; therefore, by proceeding according to the rule, we shall have, for the distance of the first point,

$$25\sqrt{5} \div 5 = 11.18034 \text{ feet.}$$

For the distance of the second point, we get

$$25\sqrt{2 \times 5} \div 5 = 15.81139 \text{ feet.}$$

For the distance of the third point, we obtain

$$25\sqrt{3 \times 5} \div 5 = 19.36492 \text{ feet,}$$

and for the distance of the fourth point, it is

$$25\sqrt{4 \times 5} \div 5 = 22.36068 \text{ feet.}$$

The preceding is all that is necessary to be calculated, for the distance of the fifth point is manifestly equal to the whole length of the parallelogram, and consequently, by the question, it is a given quantity.

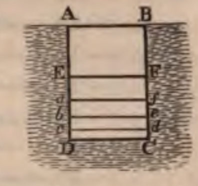
Having thus determined the distances of the several points of section below the upper surface of the fluid, the respective distances between them, or the breadths of the several sections can easily be ascertained, since they are merely the consecutive differences of the quantities above calculated; hence, we have

Distances	11.18034,	15.81139,	19.36492,	22.36068,	25.
Differences		4.63105,	3.55353,	2.99576,	2.63932;

therefore, the breadths of the respective sections, estimated in order from the surface of the fluid, are

11.18034, 4.63155, 3.55352, 2.99576, 2.63932 feet.

Let ABCD be the rectangular parallelogram, whose length AD or BC is equal to 25 feet, and the breadth AB or DC of any convenient magnitude at pleasure. Upon the length AD, set off the distances AE, Aa, Ab and Ac, respectively equal to the preceding numbers, taken in the order of their arrangement, and through the points E, a, b and c, draw the straight lines EF, af, be and cd, respectively parallel to AB or DC the horizontal sides of the parallelogram; then are the rectangular spaces AF, ae, bd and cC, the respective portions into which the given parallelogram ABCD is divided, and on which, according to the conditions of the problem, the perpendicular pressures are equal among themselves.



OF THE REQUISITE THICKNESS OF FLOOD-GATES, &c.

73. The problem which we have just resolved is a very important one; by it we can determine a limit to the requisite thickness of flood-gates and other constructions of a similar nature, and also the form which the section ought to assume, in order that the strength in every part may be proportional to the pressure sustained.

For according to the preceding notation, and by equation (7) under the third problem, the pressure on the rectangle ABFE, is

$$p = \tfrac{1}{2} b v^2 s \sin.\phi,$$

and the pressure on the whole rectangle ABCD, is

$$P = \tfrac{1}{2} b l^2 s \sin.\phi;$$

consequently, by analogy and comparison, we get

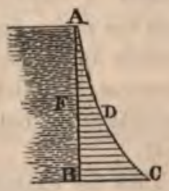
$$P : p :: l^2 : v^2.$$

And in like manner it may be shown, that the same relation obtains in respect of any other rectangle ABfa, when compared with the entire figure ABCD; consequently, the pressures are universally as the squares of the depths, the breadth being constant; therefore, the thickness should be as the square of the depth, being greatest at the bottom and decreasing upwards to the surface of the fluid.

Thus for example, let the flood-gate be of the same depth as the rectangle in the foregoing question, and let the thickness at the bottom be equal to one foot or twelve inches; then, the corresponding thicknesses for the several feet of ascent estimated upwards, will be as follows.

At 25 feet, we have $25^2 : 25^2 :: 12 : 12$ inches.
— 24 ———— $25^2 : 24^2 :: 12 : 11.060$ ——
— 23 ———— $25^2 : 23^2 :: 12 : 10.156$ ——
— 22 ———— $25^2 : 22^2 :: 12 : 9.292$ ——
— 21 ———— $25^2 : 21^2 :: 12 : 8.468$ ——
— 20 ———— $25^2 : 20^2 :: 12 : 7.680$ ——
— 19 ———— $25^2 : 19^2 :: 12 : 6.932$ ——
— 18 ———— $25^2 : 18^2 :: 12 : 6.220$ ——
— 17 ———— $25^2 : 17^2 :: 12 : 5.548$ ——
— 16 ———— $25^2 : 16^2 :: 12 : 4.856$ ——
— 15 ———— $25^2 : 15^2 :: 12 : 4.320$ ——
— 14 ———— $25^2 : 14^2 :: 12 : 3.764$ ——
— 13 ———— $25^2 : 13^2 :: 12 : 3.244$ ——
— 12 ———— $25^2 : 12^2 :: 12 : 2.764$ ——
— 11 ———— $25^2 : 11^2 :: 12 : 2.324$ ——
— 10 ———— $25^2 : 10^2 :: 12 : 1.920$ ——
— 9 ———— $25^2 : 9^2 :: 12 : 1.556$ ——
— 8 ———— $25^2 : 8^2 :: 12 : 1.228$ ——
— 7 ———— $25^2 : 7^2 :: 12 : 0.940$ ——
— 6 ———— $25^2 : 6^2 :: 12 : 0.692$ ——
— 5 ———— $25^2 : 5^2 :: 12 : 0.460$ ——
— 4 ———— $25^2 : 4^2 :: 12 : 0.308$ ——
— 3 ———— $25^2 : 3^2 :: 12 : 0.172$ ——
— 2 ———— $25^2 : 2^2 :: 12 : 0.076$ ——
— 1 ———— $25^2 : 1^2 :: 12 : 0.020$ ——

74. Having calculated the numerical values of the thicknesses or ordinates, corresponding to each foot in length, estimated from the bottom where the pressure is a maximum, upwards to the summit where it vanishes; we shall now proceed to construct the section, in order to exhibit the particular form which the preceding theory assigns.



Let the straight line A B represent the perpendicular depth of the flood-gate supporting the fluid F, and whose vertical section is denoted by the figure A B C, the exterior boundary of which is the curve line A B C, and the greatest thickness, or that at the bottom, equal to B C.

Divide the depth A B into twenty five equal parts, having an interval of one foot for each; then, through the several points of division, and parallel to the horizon, draw straight lines, beginning at the bottom and proceeding upwards, making these lines respectively equal to 12, 11.06, 10.156, 9.292, &c.

inches, according to the numbers in the foregoing tablet, and through the remote extremities of the several ordinates, trace the curve line ADC, which will mark the exterior boundary of the section.

The intelligent reader will readily perceive, that in the actual construction of the above figure, it has been found impossible to preserve the proper proportion between the several abscissæ and their corresponding ordinates; if this had been attempted, the figure must either have been enlarged to an inconvenient size, or the ordinates would have been so small as to render the general appearance of the section very indistinct.

We have therefore thought it preferable to preserve the line of the abscissæ within moderate bounds, and to enlarge the ordinates in a given constant ratio; by this means the form of the curve is correct, and the whole diagram is sufficiently distinct for practical illustration.

In all that has hitherto been done respecting the rectangular parallelogram, we have constantly considered it as being an independent plane immersed in the fluid, and having its upper side coincident with the surface; but we must now observe, that whatever relations have been shown to exist on such a supposition, the same will hold, if the plane be considered as the side of a vessel filled with the fluid by which the pressure is propagated.

We have already alluded to this principle, at the conclusion of our illustration of the fourth problem; it therefore only remains to determine by it, the pressure on the bottom and sides of a vessel filled with a fluid of uniform density, on the supposition that the bottom and the sides are respectively rectangular planes.

11. METHOD OF COMPARING THE PRESSURE ON THE PERPENDICULAR SIDES AND ON THE BOTTOM OF ANY RECTANGULAR CISTERN, BASIN, OR CANAL LOCK.

PROBLEM X.

75. Suppose that a vessel in form of a rectangular parallelopipedon, is filled with fluid of uniform density, and placed with its sides perpendicular to the horizon:—

> *It is required to compare the pressure on the upright sides with that upon the bottom, both when the sides are all equal, and when the opposite sides only are equal.*

In the solution of the present problem, it will be unnecessary to exhibit the construction of a separate diagram; because, the boundaries when considered individually, being rectangular parallelograms, the investigation for each would be similar to that required in Problem 4, and the resulting formulæ would coincide in form with that exhibited in equation (8).

Therefore, put $b =$ the horizontal breadth of the greater opposite sides,
$\beta =$ the horizontal breadth of the lesser ditto ditto,
$l =$ the perpendicular depth of the fluid, whose density is uniform,
$\mathrm{P} =$ the aggregate, or total pressure on the upright surface, and
$p =$ the pressure on the bottom.

Then, according to equation (8) under Problem 3, the pressure on each of the narrower sides is expressed by $\frac{1}{2}\beta l^2 s$, and that on each of the broader sides by $\frac{1}{2} b l^2 s$; consequently, the entire pressure on the upright surface, is

$$\mathrm{P} = l^2 s (\beta + b),$$

and by the third inference to Proposition (A), the pressure sustained by the bottom of the vessel, is

$$p = b\beta l s;$$

consequently, by analogy, we obtain

$$\mathrm{P} : p :: l(\beta + b) : \beta b.$$

Therefore, when the opposite sides of the rectangular vessel only, are equal to one another,

> *The total pressure on the upright surface, is to the pressure on the bottom, as half the area of the former is to the area of the latter.*

If $b = \beta$, or if all the sides of the vessel are equal to one another; then, the entire pressure on the upright sides, is

$$\mathrm{P} = 2 b l^2 s,$$

and that on the bottom, is

$$p = b^2 l s;$$

therefore, by analogy, we obtain

$$\mathrm{P} : p :: 2l : b.$$

Consequently, when all the four sides of the rectangular vessel are equal to one another,

> *The total pressure on the upright surface, is to the pressure on the bottom, as twice the length of the side is to its breadth.*

Again, when all the sides of the vessel have the same breadth, and the length l equal to the breadth b; then, the vessel becomes a cube, and the total pressure on the upright surface, is

$$P = 2 b^3 s,$$

and that on the bottom, is

$$p = b^3 s;$$

therefore, by analogy, we obtain

$$P : p :: 2 : 1.$$

Hence it appears, that when the vessel is a cube, that is, when the bottom and the four upright sides are equal to one another,

> *The total pressure upon the four sides, is to the pressure on the bottom, in the ratio of* 2 *to* 1.

Since the pressure on the upright surface of a cubical vessel, is double the pressure on the base; it follows, that the entire pressure which the vessel sustains, is equal to three times the pressure upon its bottom; that is,

$$P + p = 3 b^3 s. \qquad (32).$$

But the expression $b^3 s$ is manifestly equal to the weight of the fluid; consequently, the total pressure upon the sides and base of the vessel,

> *Is equal to three times the weight of the fluid which it contains.*

Now, in the case of water, where the specific gravity is represented by unity, the equation marked (32) becomes

$$P + p = 3 b^3;$$

but when the dimensions of the vessel are estimated in feet, and the pressure expressed in pounds avoirdupois, of which 62½ are equal to the weight of one cubic foot of water; then, the above equation is transformed into

$$p' = 187.5 b^3. \qquad (33).$$

Corol. This equation in its present form implies, that if the solid content of the vessel in cubic feet, be multiplied by the constant co-efficient 187.5, the product will express the number of lbs. to which the pressure on the bottom and the four upright sides is equivalent.

76. EXAMPLE 13. Suppose the length of the side of a cubical cistern to be 35 feet; what is the pressure sustained by it, when it is completely filled with water?

Here we have given, $b = 35$ feet; therefore, by proceeding according to the composition of the foregoing equation, we shall obtain

$$p' = 35 \times 35 \times 35 \times 187.5 = 8039062.5 \text{ lbs.}$$

Hence it appears, that the aggregate pressure upon the bottom and the upright surface of a cubical vessel whose side is 35 feet, is 8039062.5 lbs. or

$$8039062.5 \div 2240 = 3588.867 \text{ tons,}$$

while the absolute weight of the contained fluid, is only one third of that quantity, or

$$35 \times 35 \times 35 \times 62.5 \div 2240 = 1196.289 \text{ tons.}$$

77. EXAMPLE 14. Let the dimensions of the vessel remain as in the preceding example, and suppose it to be filled with wine, of which the specific gravity is .96, instead of water, whose specific gravity is unity, what pressure does it then sustain?

For the weight of a cubic foot of wine, we have

$$1 : 62.5 :: .96 : 60 \text{ lbs.}$$

and the pressure on the bottom and sides of a vessel containing a cubic foot is $60 \times 3 = 180$ lbs.; consequently, the pressure on the bottom and sides of a vessel whose side is 35 feet, is

$$p' = 35 \times 35 \times 35 \times 180 = 7717500 \text{ lbs.,}$$

and this, by reducing it to tons, is

$$p' = \frac{7717500}{2240} = 3445.3125 \text{ tons,}$$

while the absolute weight of the contained fluid, is only one-third of that quantity, or

$$35 \times 35 \times 35 \times 60 \div 2240 = 1148.4375 \text{ tons.}$$

CHAPTER III.

ON THE PRESSURE EXERTED BY NON-ELASTIC FLUIDS UPON PARABOLIC PLANES IMMERSED IN THOSE FLUIDS, WITH THE METHOD OF FINDING THE CENTRE OF GRAVITY OF THE SPACE INCLUDED BETWEEN ANY RECTANGULAR PARALLELOGRAM AND ITS INSCRIBED PARABOLIC PLANE.

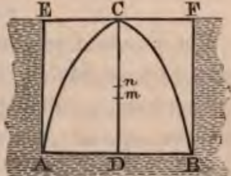
1. WHEN THE AXIS OF THE PARABOLIC PLANE IS PERPENDICULAR TO THE HORIZON, AND ITS VERTEX COINCIDENT WITH THE SURFACE OF THE FLUID.

PROBLEM XI.

77. If a parabolic plane be just perpendicularly immersed beneath the surface of an incompressible fluid:—

> *It is required to compare the pressure upon it, with that upon its circumscribing rectangular parallelogram, and to determine the intensity of pressure, according as the vertex or the base of the parabola is in contact with the surface of the fluid.*

First, when the vertex of the parabola is uppermost, and just in contact with the surface of the fluid; let ACB be the parabolic plane, of which AB is the base or double ordinate parallel to the horizon, and CD the vertical axis just covered by the fluid whose surface is EF, and let ABFE be a rectangular parallelogram circumscribing the parabola.



Now, it is demonstrated by the writers on mechanics, that the centre of gravity of a parabolic plane is situated in the vertical axis, and the point where it occurs, is at the distance of three fifths of that axis from the summit of the figure.

Therefore, if the axis CD be divided at m, into two parts such, that Cm is to Dm as 3 is to 2, then is m the centre of gravity of the parabolic space ACB; and if the axis CD be bisected in the point n; n is the centre of gravity of the circumscribing rectangular parallelogram ABEF.

Put $b = \text{AB}$, the base of the parabola, or the horizontal breadth of its circumscribing rectangular parallelogram,
$l = \text{CD}$, the vertical axis of the parabola, or the depth of its surrounding rectangle,
$d = \text{C}n$, the depth of the centre of gravity of the parallelogram ABFE below the upper surface of the fluid,
$\delta = \text{C}m$, the depth of the centre of gravity of the parabola ACB,
$\text{P} =$ the pressure on the circumscribing rectangle,
$p =$ the pressure on the parabolic plane,
$\text{A} =$ the area of the parallelogram,
$a =$ the area of the parabola, and
$s =$ the specific gravity of the fluid.

Then, according to the writers on mensuration, the area of the circumscribing rectangular parallelogram, is

$$\text{A} = bl;$$

but the area of a parabola, is equal to two thirds of the area of its circumscribing rectangle; therefore, we have

$$a = \tfrac{2}{3}bl.$$

Now, $\delta = \tfrac{3}{5}l$ by the construction, and we have shown in Proposition (A), that the pressure upon any surface,

> *Is expressed by the area of that surface, drawn into the perpendicular depth of its centre of gravity, and also into the specific gravity of the fluid.*

Consequently, the pressure perpendicular to the surface of the parabolic plane, is

$$p = \tfrac{2}{3}bl \times \tfrac{3}{5}l \times s = \tfrac{2}{5}bl^2s. \qquad (34).$$

But in order to compare the pressure on the parabola, as represented by, or implied in the above equation, with that upon its circumscribing parallelogram, we have only to recollect, that $d = \tfrac{1}{2}l$, and consequently, the pressure on the rectangle, is

$$\text{P} = bl \times \tfrac{1}{2}l \times s = \tfrac{1}{2}bl^2s;$$

consequently, by omitting the common factors, and rendering the fractions similar, we have

$$p : \text{P} :: 4 : 5.$$

78. Now, the practical rule for determining the pressure on the parabolic plane, as deduced from the equation (34), or from the preceding analogy, may be expressed in words, as follows.

RULE. *Multiply two fifths of the base of the parabola, by the square of the length of its axis drawn into the specific gravity of the fluid, and the product will express the pressure sustained by the parabolic plane, in a direction perpendicular to its surface.* Or thus:

Find the pressure on the circumscribing rectangular parallelogram, according to the second case of the rule under the third problem, and four fifths of the pressure so determined, will express the pressure perpendicular to the parabolic surface.

79. EXAMPLE 14. A parabolic plane, whose base and vertical axis are respectively equal to 28 and 42 feet, is perpendicularly immersed in a reservoir of water, so that its vertex is just in contact with the surface; what weight is equivalent to the pressure on the plane, the weight of a cubic foot of water being equal to $62\frac{1}{2}$ lbs.?

Here, according to the rule, we have

$$p = \tfrac{2}{5} \times 28 \times 42^2 \times 62\tfrac{1}{2} = 1234800 \text{ lbs.},$$

or by the second clause of the rule, it is

$$p = \tfrac{1}{2} \times 28 \times 42^2 \times 62\tfrac{1}{2} \times \tfrac{4}{5} = 1234800 \text{ lbs.}$$

Either of these methods is sufficiently simple for every practical purpose; but it will be found of essential advantage, to bear in mind the relation between the pressure on the parabola and that on its circumscribing rectangle; for which reason, the latter method may probably claim the preference.

2. METHOD OF FINDING THE CENTRE OF GRAVITY OF THE SPACE INCLUDED BETWEEN ANY RECTANGULAR PARALLELOGRAM AND ITS INSCRIBED PARABOLA.

80. It is a principle almost self-evident, that the centre of gravity, and the centre of magnitude of a rectangular parallelogram, exist in one and the same point; consequently, admitting the position of the centre of gravity of the rectangle to be known or determinable *à priori*, the position of the centre of gravity of the inscribed parabola can very readily be found.

For by knowing the position of the centre of gravity of a rectangular surface, the pressure upon it can easily be ascertained, and we have shown above, that the pressure upon a parabolic plane, and that upon the surface of its circumscribing parallelogram, are to one another in the ratio of 4 to 5; hence, when the pressure on the rectangular parallelogram is known, the pressure on the inscribed parabola is also known, being equal to four fifths of that upon the parallelogram.

Now, it has already been demonstrated, that the pressure upon any surface, whatever may be its form, is always equal to its area, drawn into the perpendicular depth of the centre of gravity, below the upper surface of the fluid; therefore, conversely, the perpendicular depth of the centre of gravity of any surface, is equal to the pressure which it sustains, divided by the area.

But from what has been demonstrated above, it is manifest that the area of the parabola and the pressure upon it, are respectively expressed by

$$\tfrac{2}{3}bl, \text{ and } \tfrac{2}{5}bl^2s;$$

consequently, by division, we obtain

$$\delta = \frac{\tfrac{2}{5}bl^2s}{\tfrac{2}{3}bl} = \tfrac{3}{5}ls,$$

and when s is expressed by unity, as in the case of water, we get

$$\delta = \tfrac{3}{5}l.$$

Now, because the parabola ACB is symmetrically divided by the axis CD, it follows, that the centre of gravity occurs in that line, and we have just shown, that it occurs at the distance of three fifths of its length from the vertex; hence, the position is determined, and that independently of computing the corresponding horizontal rectangular co-ordinate, whose intersection with the axis fixes the place of the centre sought.

The aggregate pressure upon the two equal and similar spaces AEC and BFC, is obviously equal to the difference between the pressures on the rectangular parallelogram ABFE, and that on the inscribed parabola ACD; that is,

$$p' = \mathrm{P} - p = \tfrac{1}{2}bl^2s - \tfrac{2}{5}bl^2s = \tfrac{1}{10}bl^2s;$$

where p' denotes the pressure on the spaces AEC and BFC.

Again, the area of the spaces AEC and BFC, is equal to the difference between the area of the parallelogram ABFE, and that of the inscribed parabola ACB; therefore, if a' denote the area of the triangular spaces, we have

$$a' = \mathrm{A} - a = bl - \tfrac{2}{3}bl = \tfrac{1}{3}bl.$$

But the depth of the centre of gravity of any surface, is equal to the pressure upon that surface divided by its area; consequently, the depth of the centre of gravity of the figure AECFB, which is composed of the two triangular spaces AEC and BFC, is

$$\delta' = \tfrac{1}{10} b l^2 s \div \tfrac{1}{3} b l;$$

hence, if the specific gravity of the fluid be expressed by unity, we get

$$\delta' = \tfrac{3}{10} l. \qquad (35).$$

Corol. It therefore appears, that the centre of gravity of the space, included between any rectangular parallelogram and its inscribed parabola, is situated in the axis, at the distance of three tenths of its length from the vertex.

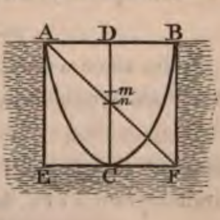
The preceding investigation determines the place of the centre of gravity to be at three tenths of the axis below a tangent line passing through the vertex of the parabola, and that it is situated in the axis is manifest; for the spaces AEC and BFC are equal, and they are similarly and symmetrically placed with respect to the axis CD; there can therefore be no reason, why the centre of gravity should occur at a point, which is nearer to the one than it is to the other; it consequently occurs at a point which is equally distant from both, and that point must obviously be found in the axis of the figure.

The above is a valuable proposition in the practice of bridge building; for by it, we can readily assign the position of the centre of gravity of the arch with all its balancing materials, and consequently, many important particulars respecting the weight and mechanical thrust, may be determined and examined with the greatest facility: all of which will be investigated and applied in our treatise on Hydraulic Architecture.

3. WHEN THE PARABOLIC PLANE PERPENDICULARLY IMMERSED HAS ITS BASE COINCIDENT WITH THE SURFACE OF THE FLUID.

81. What has hitherto been done under the present problem, applies only to the case, in which the parabola is perpendicularly immersed in the fluid, and having its vertex coincident with the surface; but when the parabolic plane is perpendicularly immersed, and having its base coincident with the fluid surface, the circumstances will be something different, as will become manifest from the following investigation.

Let ABFE be a rectangular parallelogram immersed in a fluid, with its plane perpendicular to the plane of the horizon, and having its upper side coincident with the surface of the fluid; and let ACB be a parabola described upon the rectangular plane, in such a manner, that its vertex may be downwards, its axis vertical, and its base in contact with the surface of the fluid in which it is placed.



Draw the diagonal AF intersecting CD, the vertical axis of the parabola in the point n; then is n the centre of gravity of the rectangular parallelogram ABFE; and because, as we have stated in the construction of the preceding case, the centre of gravity of a parabolic plane is situated in the axis, at the distance of three fifths of its length from the vertex; it follows, that if the axis DC be divided in the point m, into two parts such, that Dm is to mC in the ratio of 2 to 3; then shall m be the centre of gravity of the parabola ACB.

Put $b =$ AB, the horizontal breadth of the rectangular parallelogram ABFE, or the base of its inscribed parabola ACB,
$l =$ DC or AE, the vertical axis of the parabola, or the depth of its circumscribing rectangle,
$d =$ Dn, the depth of the point n, below AB the surface of the fluid,
$\delta =$ Dm, the depth of the point m as referred to AB,
$P =$ the pressure on the surface of the rectangle ABFE,
$p =$ the pressure on the parabolic surface ACB,
$A =$ the area of the rectangular parallelogram,
$a =$ the area of its inscribed parabola, and
$s =$ the specific gravity of the fluid.

Now, it is manifest from the principles of mensuration, that the area of a rectangular parallelogram, is equal to the product that arises when the two dimensions of length and breadth are multiplied into one another; that is,

$$A = bl,$$

and according to the writers on conic sections, the area of a parabola is equal to two thirds of the area of its circumscribing rectangle; therefore, we have

$$a = \tfrac{2}{3}bl.$$

Referring to the construction of the figure, we find that the axis DC is divided at m, into the two parts Dm and mC, having to one another the ratio of 2 to 3; it therefore follows, that

$$Dm = \delta = \tfrac{2}{5}l;$$

consequently, the pressure on the parabolic surface ACB, is

$$p = \tfrac{2}{3}bl \times \tfrac{2}{5}l \times s = \tfrac{4}{15}bl^2s. \qquad (36).$$

Again, since AF the diagonal of the parallelogram, bisects DC the axis of the parabola in the point n; it follows, that

$$Dn = d = \tfrac{1}{2}l;$$

therefore, the pressure perpendicular to the surface of the rectangular parallelogram ABFE, becomes

$$P = bl \times \tfrac{1}{2}l \times s = \tfrac{1}{2}bl^2s;$$

hence, by analogy, we obtain

$$p : \mathrm{P} :: \tfrac{4}{15} b l^2 s : \tfrac{1}{2} b l^2 s,$$

and this, by omitting the common quantities, b, l^2 and s, becomes

$$p : \mathrm{P} :: \tfrac{4}{15} : \tfrac{1}{2},$$

and finally, by reducing the fractions in this analogy to a common denominator, we shall obtain

$$p : \mathrm{P} :: 8 : 15.$$

82. Having thus established the formula for determining the pressure on the parabolic plane, and also the ratio which compares the said pressure with that upon the circumscribing rectangular parallelogram; we shall in the next place deduce the rules, by which the numerical process is to be performed, when the principle is applied to the actual determination of the pressure in reference to cases of practice. The rule, as derived from the equation numbered (36), may be expressed in the following manner.

> RULE. *Multiply the base of the parabola by the square of its vertical axis, and again by the specific gravity of the fluid; then, four fifteenths of the product will express the pressure perpendicular to the surface of the parabolic plane.*

The rule for determining the pressure on the surface of a parabola, as deduced from the analogy of comparison investigated above, may be expressed in words in the following manner.

> *Find the pressure perpendicular to the surface of the circumscribed rectangular parallelogram, after the manner described in the second case of the rule under the third problem; then, eight fifteenths of the pressure so determined, will express the pressure on the parabolic plane.*

83. EXAMPLE 15. The data remaining as in the preceding case; what will be the pressure on the parabolic plane, its axis being vertical, and its base in contact with the surface of the fluid?

$$28 \times 42 \times 42 \times 62\tfrac{1}{2} = 3087000,$$

and four fifteenths of this, is

$$3087000 \times 4 \div 15 = 823200 \text{ lbs.}$$

This result is derived from the first of the above rules, or that which corresponds to the equation (36); and the process as performed by the second rule, or that obtained from the ratio of comparison, will stand as below.

$$\tfrac{1}{2} \times 28 \times 42 \times 42 \times 62\tfrac{1}{2} \times \tfrac{8}{15} = 823200 \text{ lbs.}$$

Hence, the pressure on the plane in this case, is only two thirds of what we found it to be in the foregoing case, where the vertex is in contact with the surface of the fluid.

With respect to the position of the centre of gravity in this case, it is manifest, that the mode of discovering it, is similar to that which we employed in the case immediately preceding, where the axis of the parabola was supposed to be vertical, and its summit in contact with the surface of the fluid; it is therefore unnecessary to repeat the investigation, but there is another condition of the figure remaining to be considered, in which a knowledge of the position of the centre of gravity becomes of more importance, as will readily appear from the circumstances which present themselves in the solution of the following problem.

4. WHEN THE BASE OF THE PARABOLIC PLANE IS PERPENDICULAR TO THE HORIZON, ITS AXIS HORIZONTAL, AND THE PRESSURE UPON IT IS TO BE DETERMINED AS COMPARED WITH THAT UPON ITS CIRCUMSCRIBING RECTANGULAR PARALLELOGRAM.

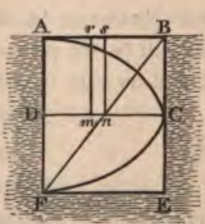
PROBLEM XII.

84. If a parabolic plane be perpendicularly immersed in an incompressible fluid, in such a manner, that its base may be vertical, and just in contact with the surface:—

> *It is required to determine the pressure upon it, and to compare it with that upon its circumscribing rectangular parallelogram.*

Let ABEF be a rectangular parallelogram immersed in a fluid, with its plane perpendicular to the plane of the horizon, and its upper side AB coincident with the surface of the fluid in which it is immersed.

Bisect AF and BE in the points D and C; join DC, and upon AF as a base, with the corresponding axis DC, describe the parabola ACF, touching AB the surface of the fluid in the point A; then is ACF the surface for which the pressure is required to be investigated.



Join BF, intersecting DC the axis of the parabola in the point *n*; then is *n* the centre of gravity of the rectangular parallelogram ABEF.

Divide the axis DC into two parts Dm and Cm such, that Dm is to Cm, in the ratio of 2 to 3; and the point m thus determined, is the place of the centre of gravity of the parabola ACF. Through the points m and n, draw the straight lines mr and ns respectively perpendicular to the axis DC; then are mr and ns, the perpendicular depths of the points m and n below AB, the surface of the fluid, and which in the present case are equal to one another.

Put $b =$ AB, the horizontal breadth of the rectangular parallelogram ABEF, or the axis of its inscribed parabola ACF,
$l =$ AF, the length of the circumscribing rectangle, or the base of the inscribed parabola,
$d = rm$ or sn, the vertical depths of the centres of gravity, below AB the surface of the fluid,
$\mathrm{P} =$ the pressure on the rectangle ABEF,
$p =$ that on the inscribed parabola ACF,
$\mathrm{A} =$ the area of the circumscribing rectangular parallelogram,
$a =$ the area of the parabola, and
$s =$ the specific gravity of the fluid in which they are immersed.

Then, according to the principles of mensuration, the area of the rectangular parallelogram ABEF, is equal to the product of the breadth AB drawn into the depth AF; that is,

$$\mathrm{A} = bl;$$

and by the property of the parabola, its area is

$$a = \tfrac{2}{3} b\, l.$$

But the pressure perpendicular to the surface of the rectangular parallelogram, is, as we have already frequently stated, expressed by the area drawn into the perpendicular depth of the centre of gravity; and this being the case, whatever may be the form of the surface pressed, it follows, that the pressure on the rectangle ABEF, is

$$\mathrm{P} = bl \times d \times s = bdls;$$

and that on the parabola ACF is

$$p = \tfrac{2}{3} bl \times d \times s = \tfrac{2}{3} bdls.$$

Now, it is manifest from the nature of the figure, and from the principles upon which it is constructed, that rm and sn are each of them equal to $\tfrac{1}{2}$AF; that is, $d = \tfrac{1}{2} l$; therefore, let $\tfrac{1}{2} l$ be substituted for d in each of the above equations, and we shall obtain

For the rectangle ABEF,

$$\mathrm{P} = \tfrac{1}{2} b l^2 s;$$

and for the parabola ACF, it is

$$p = \tfrac{1}{3} b\, l^2 s. \qquad (37).$$

Consequently, by analogy, the comparative pressures on the parabola and its circumscribing rectangle, are as follows:

$$p : \mathrm{P} :: \tfrac{1}{3} b l^2 s : \tfrac{1}{2} b l^2 s;$$

and from this, by casting out the common quantities and assimilating the fractions, we get

$$p : \mathrm{P} :: 2 : 3.$$

Corol. Hence it appears, that when the axis of the parabola is horizontal, and its base perpendicular to the horizon; the pressure perpendicular to its surface, when compared with that on its circumscribing parallelogram, bears precisely the same relation, that its area bears to the area of the rectangle by which it is circumscribed.

85. The practical rule for determining the pressure on the parabolic plane, when placed in the position specified in the problem, may be expressed in words at length in the following manner.

> Rule. *Multiply the horizontal axis, by the square of the vertical base or double ordinate, and again by the specific gravity of the fluid; then, take one third of the product for the pressure perpendicular to the surface of the parabolic plane.*
>
> Or thus, *Calculate the pressure on the circumscribing rectangle, and take two thirds of the result for the pressure on the parabola.*

86. Example 16. The data remaining as in the example to the foregoing problem, it is required to determine the pressure on the parabolic plane, supposing its axis to be horizontal, its base or double ordinate vertical, and the upper extremity of the base in contact with the surface of the fluid, which, according to the conditions of the previous question, is *water*, whose specific gravity is expressed by unity, and the weight of one cubic foot of which is equal to 62½ lbs. *avoirdupois?*

Referring the numerical data to the same parts of the figure, as in the preceding cases, we have given $b = 42$ feet; $l = 28$ feet, and $s = 62\frac{1}{2}$ lbs.; therefore, by proceeding according to the rule, we have

$$42 \times 28 \times 28 \times 62\tfrac{1}{2} = 2058000,$$

which being divided by 3, gives

$$p = 2058000 \div 3 = 686000 \text{ lbs.}$$

Hence it appears, that the total pressures perpendicular to the parabolic surface, according to the several positions in which we have placed it, are to one another respectively as the numbers

3087, 2058 and 1715;

the first two of which, by reason of the parts of the figure being the same in each, are obviously dependent upon one another; but the third, in which the parts of the figure are reversed, is wholly independent and distinct from the other two.

87. Corol. 1. Admitting therefore, that the pressure upon the parabolic surface, under the three circumstances of position in which we have considered it, is represented by the equations (35), 36) and (37); it follows, that the situation of the centre of gravity can easily be ascertained; for the pressure in each case, as we have elsewhere shown, is represented by the area of the figure, drawn into the perpendicular depth of its centre of gravity; consequently by reversing the process, the depth of the centre of gravity will become known, if the pressure be divided by the area of the surface on which the fluid presses.

Corol. 2. Since the parabola is a figure symmetrically situated with respect to its axis, it is obvious, that the centre of gravity of its surface must occur at the same point, in whatsoever position it may be placed; but when the place of its centre is referred to the surface of the fluid in which it is immersed, the distance varies for each particular case: thus,

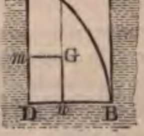
In the first instance, the perpendicular distance, is $= \frac{3}{5}$ths of the axis,
—— second,———————————— $= \frac{2}{5}$ths ————
—— third, ———————————— $= \frac{1}{2}$ the base.

But as we have just stated, the centre of gravity of the parabolic surface as referred to its vertex, or any other fixed point, in all these cases, remains unaltered, in whatever position the figure itself may be placed.

5. THE METHOD OF DETERMINING THE PRESSURE OF THE FLUID UPON A SEMI-PARABOLIC PLANE AS COMPARED WITH THAT ON THE CIRCUMSCRIBING RECTANGULAR PARALLELOGRAM.

88. When the semi-parabola only is considered, the determination of its centre of gravity, and consequently, of the pressure on its surface becomes more difficult; for, since the figure is not symmetrical with respect to its axis, we are under the necessity of computing the two rectangular co-ordinates, whose intersection determines the place of the required centre.

Let CBD be a semi-parabola, perpendicularly immersed in a fluid, so that its axis CD is vertical, and the vertex in contact with CF the surface of the fluid, and let CFBD be the circumscribing rectangular parallelogram.



Now, since by Problem XI, it has been proved that the pressure on the entire parabola with its axis vertical, is equal to four fifths of that upon its circumscribing parallelogram; it follows, that the pressure on the semi-parabola in the same position, is also equal to four fifths of that upon its circumscribing parallelogram, it being manifestly equal to half the pressure on the whole parabola.

Divide the axis CD into two parts, such, that Dm and Cm shall be to one another in the ratio of 2 to 3; and in like manner, let the ordinate or base BD be divided into two parts, such, that Dn and Bn shall be to one another in the ratio of 3 to 5*; then, through the points m and n, and respectively parallel to DB and DC, draw the straight lines mG and nG, meeting each other in G, the centre of gravity of the semi-parabola DCB.

Put $b =$ CF or DB, the horizontal breadth of the rectangle CFBD, or the base of the semi-parabola CBD,
$l =$ CD or FB, the vertical depth of the rectangle, or axis of its inscribed semi-parabola,
$d =$ Cm or EG, the perpendicular depth of the centre of gravity,
$P =$ the pressure on the rectangular parallelogram CFBD,
$p =$ the pressure on its inscribed semi-parabola,
$A =$ the area of the parallelogram,
$a =$ the area of the semi-parabola, and
$s =$ the specific gravity of the fluid in which they are immersed.

Then, according to the principles of mensuration, the area of a rectangle is expressed by the product of its two dimensions; that is, by its length drawn into its breadth; therefore, we have

$$A = bl,$$

and by Proposition (A), the pressure exerted by a fluid, perpendicularly to any surface immersed in it, or otherwise exposed to its influence,

> *Is equal to the area of the surface pressed, drawn into the perpendicular depth of its centre of gravity, and again into the specific gravity of the fluid.*

Consequently, the pressure on the circumscribing rectangular parallelogram CFBD, becomes

$$P = bl \times \tfrac{1}{2}l \times s = \tfrac{1}{2}bl^2s.$$

---

* It is demonstrated by the writers on mechanics, that the centre of gravity of a semi-parabola is situated in its plane, at the distance of three eighths of the ordinate from the axis, and two fifths of the axis from the ordinate, or three fifths of the axis from its vertex.

Now, since the pressure on the semi-parabola is equal to four fifths of the pressure on its circumscribing parallelogram, we shall obtain

$$\tfrac{4}{5}P = \tfrac{4}{5} \times \tfrac{1}{2}bl^2s = \tfrac{2}{5}bl^2s = p. \qquad (38).$$

The expression which we have here determined for the pressure on the surface of the semi-parabola DCB, is precisely the same as that which we have given in equation (35) for the entire figure; only in the present instance, the value of b, the horizontal breadth of the parallelogram, is but one half the value as applied to the parabola, when placed under the conditions specified in the eleventh problem foregoing.

89. If the symbol b retain its former value, that is, if it be referred to the base of the entire parabola, or to the breadth of the parallelogram circumscribing the entire parabola, then, the pressure on the semi-parabola, becomes

$$p = \tfrac{1}{5}bl^2s. \qquad (39).$$

Consequently, the practical rule for determining the pressure on the semi-parabola as deduced from this equation, may be expressed as follows.

> RULE. *Multiply one fifth of the base, or double ordinate of the whole parabola, by the square of the length of its axis, and again by the specific gravity of the fluid, and the product will express the pressure on the semi-parabola in a direction perpendicular to its surface.*

But if the symbol b refer to the ordinate, or base of the semi-parabola, then, the rule as deduced from the equation (38), will be precisely the same as that which we have given under the equation numbered (35) in Problem XI, to which place the reader is referred for the purpose of avoiding a direct repetition.

90. EXAMPLE 17. A plane in the form of a semi-parabola whose base or ordinate is 16 feet, and its axis 40 feet, is perpendicularly immersed in a cistern of water, in such a manner, that its axis is vertical, and its vertex in contact with the surface of the fluid; what pressure does it sustain, the weight of a cubic foot of water being equal to 62½ lbs.?

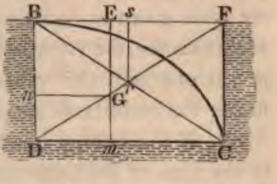
The equation in its present state, supposes the ordinate, or base of the semi-parabola to be given, and therefore, the pressure is determined by the rule to the equation (35) or (38), in the following manner:

$$p = \tfrac{2}{5} \times 16 \times 40^2 \times 62\tfrac{1}{2} = 640000 \text{ lbs.}$$

But in order to determine the pressure by the rule immediately preceding, we must suppose the breadth or base of the figure to be doubled, in which case it will have reference to the whole parabola, and the pressure will be reduced to that upon its half, by employing the constant $\frac{1}{5}$ instead of $\frac{2}{5}$ according to the rule, thus,

$$p = \tfrac{1}{5} \times 32 \times 40^2 \times 62\tfrac{1}{2} = 640000 \text{ lbs.}$$

91. If the axis of the semi-parabola were horizontal and its ordinate vertical, as in the annexed diagram; then, the area of the semi-parabolic figure, as well as that of its circumscribing parallelogram, will remain the same, but the pressures perpendicular to their respective surfaces will be very different.



Divide BD in n, in such a manner, that Bn and Dn may be to one another in the ratio of 5 to 3; and in like manner, divide CD in m, so that Cm and Dm shall be to each other in the ratio of 3 to 2.

Through the points n and m, and parallel respectively to BF and BD, draw the straight lines nG and mG, intersecting one another in the point G, and produce mG to E; then, by the note to the preceding case, the point G is the centre of gravity of the semi-parabola DCB, and EG is its perpendicular depth below BF, the horizontal surface of the fluid.

Therefore, let the preceding notation remain, and let the several symbols refer to the same parts of the figure as in the preceding case, disregarding the change of position which has taken place; then, as before, the area of the parallelogram BFCD, is

$$\text{A} = bl,$$

and the pressure perpendicular to its surface, is

$$\text{P} = \tfrac{1}{2}b^2 ls.$$

For draw the diagonals BC and FD intersecting each other in the point r, and through r draw rs parallel to BD or Em; then, r is the centre of gravity of the rectangle BFCD, and sr its perpendicular depth below BF the surface of the fluid; but according to our notation $sr = \frac{1}{2}b$, and we have seen above, that $\text{A} = bl$; now, the pressure on any surface, whatever may be its form,

> *Is equal to the product that arises, when the area of the surface pressed, is drawn into the perpendicular depth of its centre of gravity, and again, into the specific gravity of the fluid.*

Consequently, the pressure perpendicular to the surface of the rectangular parallelogram BFCD, is

$$\text{P} = bl \times \tfrac{1}{2}b \times s = \tfrac{1}{2}b^2 ls.$$

Again, the area of the semi-parabola BCD, is equal to two thirds of its circumscribing rectangular parallelogram BFCD; therefore we have

$$a = \tfrac{2}{3} \times bl = \tfrac{2}{3}bl,$$

and the pressure perpendicular to its surface, is

$$p = \tfrac{5}{12}b^2 l\,s.$$

This is manifest, for according to the construction and the nature of the figure of the parabola, B*n* or EG is equal to five eighths of BD; therefore, we have

$$p = \tfrac{2}{3}bl \times \tfrac{5}{8}b \times s = \tfrac{5}{12}b^2 ls; \qquad (40).$$

consequently, by analogy, we obtain

$$p : \mathrm{P} :: \tfrac{5}{12}b^2 ls : \tfrac{1}{2}b^2 ls.$$

Therefore, by suppressing the common factors, and rendering the fractions $\tfrac{5}{12}$ and $\tfrac{1}{2}$ similar, we shall get

$$p : \mathrm{P} :: 5 : 6;$$

hence it appears, that when the ordinate of the semi-parabola is vertical, and its upper extremity in contact with the surface of the fluid:—

> *The pressure upon the semi-parabola, is to that upon its circumscribing rectangular parallelogram, as* 5 *is to* 6, *or as* 1 *is to* 1.2.

92. Consequently, the practical rule for determining the pressure in the present instance, as deduced from the equation marked (40), or from the above analogy, may be expressed as follows.

> RULE. *Multiply the square of the given ordinate by the axis of the semi-parabola, and again by the specific gravity of the fluid; then, five twelfths of the result will give the pressure sought.* Or thus,
>
> *Find the pressure on the circumscribing parallelogram, and take five sixths of the pressure thus found, for the pressure on the semi-parabola.*

93. EXAMPLE 18. Let the numerical values of the axis and ordinate remain as in the preceding example; what will be the pressure on the surface of the semi-parabola, supposing the axis to be horizontal, the ordinate vertical, and its remote extremity in contact with the surface of the fluid?

If the operation be performed according to the rule deduced from equation (40), we shall obtain

$$p = 16^2 \times 40 \times 62\tfrac{1}{2} \times \tfrac{5}{12} = 266666\tfrac{2}{3} \text{ lbs.}$$

but if the operation be performed according to the rule derived from the analogy of comparison, we shall have

$$p = \tfrac{5}{6}P; \text{ that is, } p = \tfrac{1}{2} \times 16^2 \times 40 \times 62\tfrac{1}{2} \times \tfrac{5}{6} = 266666\tfrac{2}{3} \text{ lbs.}$$

6. THE METHOD OF DETERMINING THE POSITION OF THE CENTRE OF GRAVITY OF THE SPACE COMPREHENDED BETWEEN THE PARABOLIC CURVE AND ITS CIRCUMSCRIBING PARALLELOGRAM.

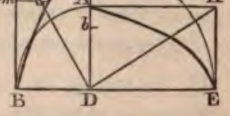
94. By means of the pressure on the semi-parabola, as we have investigated it in the two foregoing cases, we are enabled to determine the pressure on the space CFB, and from thence, the position of its centre of gravity.

This is an important inquiry in the practice of bridge building, for, in determining the thickness of piers necessary to resist the drift or shoot of a given arch, independently of the aid afforded by the other arches, it becomes requisite to find the centre of gravity of the spandrel or space BFC, which is used for the purpose of balancing the arch and filling up the haunches or flanks.

Now, the method which has generally been employed for the determination of this centre is extremely operose, and in many cases it involves considerable difficulty, requiring the calculations of solids and planes, which are by no means easy; but the method which we are about to employ, requires no such tedious and prolix operations, as will become manifest from the following investigation, which refers to the space comprehended between a semi-parabola and its circumscribing rectangle.

Let BCD be a semi-parabola, having the axis DC vertical while the corresponding ordinate DB is horizontal, and let BDCF be the circumscribing rectangular parallelogram.

Suppose the point D to remain fixed, and conceive the semi-parabola BCD to revolve about the point D until it comes into the position AED, where the axis DE is horizontal, and the corresponding ordinate DA vertical; then it is manifest, that the circumscribing rectangular parallelogram ADEH in this latter position, is equal to BDCF in the former, and consequently, the space AHE comprehended between the sides of the rectangle AH, HE and the curve AE, is equal to the space EFC similarly constituted.



It is further manifest, that while the semi-parabola revolves about the point D, from the position BCD to that of AED, the points B and C, the extremities of the ordinate and axis, describe respectively, the circular quadrants BA and CE, while the point F describes another quadrant, whose containing radii are the diagonals DF and DH.

Put $b =$ BD or AD, the ordinate of the semi-parabola in either position,
$l =$ CD or ED, the corresponding axis,
$d = n$G, the depth of the centre of gravity of the space BFC, when the axis is vertical,
$\delta = m$G, or Ab, the depth of the centre of gravity of the space AHE or BFC, when the axis is horizontal,
$\mathrm{P} =$ the pressure on the circumscribing rectangular parallelogram BDCF, or AHED,
$p =$ the pressure on the inscribed semi-parabola, and
$p' =$ the pressure on the space comprehended between the semi-parabola and its circumscribing rectangular parallelogram.

Then, according to equation (8) under the third problem, the pressure on the circumscribing rectangular parallelogram when the length is vertical, is

$$\mathrm{P} = \tfrac{1}{2} b l^2 s;$$

and agreeably to equation (38) under the eleventh problem, the pressure on the inscribed semi-parabola with the axis vertical, is

$$p = \tfrac{2}{5} b l^2 s;$$

consequently, by subtraction, the pressure upon the space BFC, comprehended between the sides of the parallelogram BF, FC and the curve of the parabola BC, is

$$p' = \mathrm{P} - p = \tfrac{1}{2} b l^2 s - \tfrac{2}{5} b l^2 s;$$

hence, by suppressing the symbol for the specific gravity, we get

$$p' = b l^2 (\tfrac{1}{2} - \tfrac{2}{5}) = \tfrac{1}{10} b l^2. \qquad (41).$$

Now, according to the writers on mensuration, the area of the semi-parabola is equal to two thirds of the area of the circumscribing parallelogram; it therefore follows, that the area of the space BFC, is equal to one third of the rectangle BDCF; that is,

$$bl - \tfrac{2}{3} bl = \tfrac{1}{3} bl.$$

But it has been demonstrated, that the pressure upon any surface, is equal to the area of that surface, drawn into the perpendicular depth of the centre of gravity; consequently, we have

$$p' = \tfrac{1}{3}bdl;$$

and we have shown above in equation (41), that when the axis of the semi-parabola is vertical, the pressure on the space BFC is

$$p' = \tfrac{1}{10}bl^2;$$

consequently, by comparison, we obtain

$$\tfrac{1}{3}bdl = \tfrac{1}{10}bl^2,$$

and finally, dividing by $\frac{1}{3}bl$, we shall have

$$d = \tfrac{1}{10}bl^2 \div \tfrac{1}{3}bl = \tfrac{3}{10}l. \qquad (42).$$

Again, when the axis of the semi-parabola is horizontal, as indicated by AED, the pressure on the circumscribing rectangle, according to equation (10) under the third problem, is

$$\mathrm{P} = \tfrac{1}{2}b^2ls,$$

and the pressure upon the inscribed parabola, according to equation (40) under the eleventh problem, is

$$p = \tfrac{5}{12}b^2ls;$$

therefore, by subtraction, the pressure upon the space comprehended between the rectangular parallelogram and its inscribed semi-parabola, is

$$p' = \mathrm{P} - p = \tfrac{1}{2}b^2ls - \tfrac{5}{12}b^2ls,$$

and by suppressing the symbol for the specific gravity, we have

$$p' = \tfrac{1}{12}b^2l.$$

Now, the area of the inscribed semi-parabola is, as we have seen above, equal to two thirds of its bounding rectangle, and the area of the space comprehended between the rectangle and the semi-parabola, is therefore, equal to one third of the same quantity; that is,

$$bl - \tfrac{2}{3}bl = \tfrac{1}{3}bl;$$

consequently, the pressure on the irregular space AHE, is

$$p' = \tfrac{1}{3}b\delta l;$$

hence, by comparison, we shall have

$$\tfrac{1}{3}b\delta l = \tfrac{1}{12}b^2l;$$

therefore, by division, we obtain

$$\delta = \tfrac{1}{12}b^2l \div \tfrac{1}{3}bl = \tfrac{1}{4}b. \qquad (43).$$

Having thus determined the values of the rectangular co-ordinates, as represented by the equations (42) and (43), the position of the centre of gravity can easily be found; for, from the point F, set off Fm equal to three tenths of the axis CD, and Fn equal to one fourth of the ordinate BD, or its equal FC; then, through the points m and n, parallel respectively to the ordinate BD and axis CD, draw the straight lines *m*G and *n*G, intersecting each other in the point G; and the point G thus determined, is the position of the centre of gravity of the space comprehended between the semi-parabola and its circumscribing rectangular parallelogram.

This method of determining the position of the centre of gravity of the space comprehended between the curve and its circumscribing parallelogram, will be illustrated and applied in all its generality, when we come to treat on the subject of Hydraulic Architecture, to which it more properly belongs; and for this reason, we shall take no further notice of it in this place, but proceed with our inquiry respecting pressure, which is more immediately the object of our research.

7. METHOD OF DIVIDING A PARABOLIC PLANE PARALLEL TO ITS BASE, SO THAT THE FLUID PRESSURES ON EACH PART MAY BE EQUAL TO ONE ANOTHER.

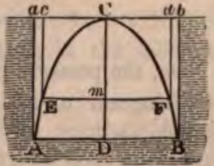
PROBLEM XIII.

95. If a parabolic plane be immersed perpendicularly in an incompressible fluid, in such a manner, that its vertex is just in contact with the surface :—

> *It is required to determine at what distance from the vertex, a straight line must be drawn parallel to the base, so that the figure may be divided into two parts, on which the pressures are equal to one another.*

Let ACB be the given parabola, immersed in the fluid after the manner specified in the problem, and let *a*AB*b* be the circumscribing rectangular parallelogram.

Take C*m* for the distance from the vertex through which the line of division passes, and draw EF parallel to the base AB; then are the spaces ABFE and ECF, the parts into which the parabola is divided, and on which, by the conditions of the problem, the pressures are equal.



Through the points E and F, the extremities of the line of division, draw E*c* and F*d* respectively parallel to CD the axis of the figure; then is *c*EF*d*, the rectangular parallelogram circumscribing the parabola ECF.

Put $2b =$ AB or ab, the base of the parabola ACB, or the horizontal breadth of the circumscribing parallelogram $a$AB$b$,
$l =$ CD or aA, the vertical axis of the parabola ACB, or the depth of the rectangle by which it is encompassed,
$2y =$ EF or Cd, the base of the parabola ECF, or the breadth of the rectangle CEFd,
$x =$ Cm or CE, the axis of the parabola ECF, or the depth of its circumscribing rectangle CEFd,
$P =$ the pressure on the rectangular parallelogram $a$AB$b$, circumscribing the parabola ACB,
$p =$ the pressure on the inscribed parabola,
$P' =$ the pressure on the rectangular parallelogram CEFd, circumscribing the parabola ECF,
$p' =$ the pressure on the inscribed parabola, and
$s =$ the specific gravity of the fluid.

Then since AB $= 2b$ and EF $= 2y$, it follows, that AD $= b$ and E$m = y$; therefore, by the property of the parabola, we have

$$l : b^2 :: x : y^2;$$

and consequently, by equating the products of the extreme and mean terms, we shall obtain

$$ly^2 = b^2 x,$$

and by division, it is

$$y^2 = \frac{b^2 x}{l},$$

and this, by extracting the square root, becomes

$$y = b\sqrt{\frac{x}{l}},$$

and finally, multiplying by 2, we obtain

$$\text{EF} = 2y = 2b\sqrt{\frac{x}{l}},$$

Now, the pressure perpendicular to the surface of the rectangular parallelogram $a$AB$b$, according to equation (8) under the third problem, is

$$P = b l^2 s;$$

but we have seen elsewhere, that the pressure on the surface of a parabola, is equal to four fifths of that upon its circumscribing parallelogram; consequently, the pressure on the parabola ACB, is

$$p = \tfrac{4}{5} b l^2 s, \qquad (44).$$

Again, the pressure perpendicular to the surface of the rectangular parallelogram CEF*d*, is

$$P' = bx^2s\sqrt{\frac{x}{l}},$$

and the pressure upon the inscribed parabola ECF, is four fifths of the pressure on the circumscribing rectangle; that is,

$$p' = \tfrac{4}{5}bx^2s\sqrt{\frac{x}{l}}; \qquad (45).$$

but according to the conditions of the problem,

$p' = \frac{1}{2}p$; hence we have

$$\tfrac{4}{5}bx^2s\sqrt{\frac{x}{l}} = \tfrac{2}{5}bl^2s,$$

and by suppressing the common factors, we get

$$x^2\sqrt{\frac{x}{l}} = \tfrac{1}{2}l^2,$$

from which, by squaring both sides, we obtain

$$x^5 = \tfrac{1}{4}l^5;$$

consequently by extracting the fifth root of both sides, we get

$$x = l\sqrt[5]{\tfrac{1}{4}};$$

but according to the arithmetic of surd quantities

$$\sqrt[5]{\tfrac{1}{4}} = \tfrac{1}{2}\sqrt[5]{8};$$

therefore, by substitution, we shall have

$$x = \tfrac{1}{2}l\sqrt[5]{8};$$

now, the sursolid, or fifth root of 8, is 1.51571; hence we get

$$x = .75785l. \qquad (46).$$

96. The practical rule supplied by this equation is extremely simple; it may be expressed in words at length in the following manner.

> **Rule.** *Multiply the axis of the given parabola by the constant number* .75785, *and the product will give the distance from the vertex through which the line of division passes.*

97. **Example** 19. The axis of a parabola is 29 feet, and its plane is perpendicularly immersed in a cistern of water, in such a manner, that its vertex is just in contact with the surface; through what point

in the axis must a line be drawn parallel to the base, so that the pressures on the two parts into which the parabola is divided, may be equal to one another?

By operating according to the preceding rule derived from equation (46), we shall have for the distance from the vertex

$$x = .75785 \times 29 = 21.97765 \text{ feet.}$$

In the case which we have investigated above, the parabola is divided into two parts on which the pressures are equal; but in order to render the solution general, we must so arrange it, that the parts of division may bear any ratio to one another, as denoted by the symbols m and n; that is,

$$p : p - p' :: m : n.$$

Now, we have seen in equation (44), that the pressure on the entire parabola ACB, is

$$p = \tfrac{4}{5} b l^2 s,$$

and according to equation (45), the pressure on the parabola ECF, is

$$p' = \tfrac{4}{5} b x^2 s \sqrt{\frac{x}{l}});$$

but the pressure upon the space AEFB, is manifestly equal to the difference of these; that is,

$$\tfrac{4}{5} b l^2 s - \tfrac{4}{5} b x^2 s \sqrt{\frac{x}{l}} = \tfrac{4}{5} b s (l^2 - x^2 \sqrt{\frac{x}{l}});$$

consequently, we obtain

$$x^2 \sqrt{\frac{x}{l}} : (l^2 - x^2 \sqrt{\frac{x}{l}}) :: m : n;$$

and from this, by equating the products of the extreme and mean terms, we shall obtain

$$n x^2 \sqrt{\frac{x}{l}} = m (l^2 - x^2 \sqrt{\frac{x}{l}}),$$

which, by transposing and collecting the terms, becomes

$$(m + n) x^2 \sqrt{\frac{x}{l}} = m l^2,$$

from which, by division, we shall get

$$x^2 \sqrt{\frac{x}{l}} = \frac{m l^2}{m + n};$$

therefore, by involving or squaring both sides of this equation, we shall obtain

$$x^5 = \frac{m^2 l^5}{(m+n)^2};$$

and by extracting the sursolid root, we get

$$x = l\sqrt[5]{\frac{m^2}{(m+n)^2}}. \qquad (47).$$

If m and n be equal to one another as in the preceding case, then it is obvious that the equation becomes

$$x = l\sqrt[5]{\tfrac{1}{4}},$$

and if any other numerical ratio be proposed, such as 4 to 5; then we shall have

$$x = l\sqrt[5]{\tfrac{16}{81}}.$$

Thus for example; let the axis of the parabola remain as in the foregoing question, and let it be required to find a point, through which a line must be drawn parallel to the base, so that the pressure on the part above the dividing line, may be to that below it in the ratio of 4 to 5?

Here we have

$$x = 29\sqrt[5]{\tfrac{16}{81}} = 20.967 \text{ feet}.$$

Hence it appears, that if a point be taken in the axis of the given parabola at the distance of 20.967 feet from the vertex, and if through that point, a line be drawn parallel to the base, the parabola will be divided into two parts, on which the pressures are to one another as 4 to 5.

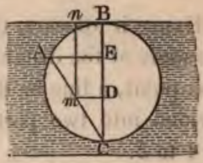
CHAPTER IV.

OF THE PRESSURE OF INCOMPRESSIBLE FLUIDS ON CIRCULAR PLANES AND ON SPHERES IMMERSED IN THOSE FLUIDS,—THE EXTREMITY OF THE DIAMETER OF THE FIGURE BEING IN EACH CASE IN EXACT CONTACT WITH THE SURFACE OF THE FLUID.

PROBLEM XIV.

98. Suppose a circular plane to be immersed perpendicularly in an incompressible fluid, in such a manner, that the extremity of the diameter is just in contact with the surface:—

> ***It is required to draw from the lowest point of the circular plane, that chord on which the pressure shall be a maximum.***



Let ABC be the circular plane immersed in the fluid according to the conditions of the problem; draw the vertical diameter BC touching the surface of the fluid in the point B, and let CA be the chord required.

Bisect the chord CA in the point m, and through the point m thus determined, draw mn parallel to BC the vertical diameter, meeting the surface of the fluid in n; then is nm the perpendicular depth of the centre of gravity of the chord AC, below the surface of the fluid in which it is immersed; draw also AE and mD respectively perpendicular to the diameter BC.

Now, we have already demonstrated in the first Problem, that the pressure upon a physical line, is equal to the product of its length by the perpendicular depth of its centre of gravity, and again by the specific gravity of the fluid; consequently, we have

$$p = \text{AC} \times nm \times s.$$

Put $d = \text{BC}$, the diameter of the immersed circular plane,

$\delta = mn$, the perpendicular depth of the centre of gravity of the chord AC,

$l =$ AC, the length of the chord on which the pressure is a maximum,
$p =$ the pressure on the chord AC,
$x =$ CD, the perpendicular height of the centre of gravity of the chord AC above the lower extremity of the diameter,
and $s =$ the specific gravity of the fluid.

Then we have $\delta = d - x$; $\text{CE} = 2x$, and by the property of the circle, the length of the chord becomes

$$l = \sqrt{2dx};$$

consequently, the pressure upon it is

$$p = (d - x)\, s \sqrt{2dx};$$

but this, according to the conditions of the problem, is to be a maximum; therefore, by putting the fluxion of the expression equal to nothing, we obtain

$$2d\dot{x}\,(d^2 - 4dx + 3x^2) = 0;$$

therefore, by omitting the common factors $2d\dot{x}$ and transposing, we shall have

$$3x^2 - 4dx = -d^2,$$

and this quadratic equation being reduced, we get

$$x = \tfrac{1}{3}d. \qquad (48).$$

Corol. 1. Consequently, to determine the chord by construction, make BE equal to one third of the vertical diameter BC, and through the point E, draw the straight line EA at right angles to BC, and meeting the circumference in the point A; then from C, the lower extremity of the diameter, inflect the straight line CA, and the thing is done. Or thus:

2. Find ϕ an angle such, that $\tan.\,\phi = \frac{1}{2}\sqrt{2} = .70711$, which happens when $\phi = 35^\circ\ 15'\ 51''$; therefore, at C the lower extremity of the diameter, make the angle BCA equal to 35° 15′ 51″, and the straight line CA will be the chord required.

It has been shown above, that according to the property of the circle, the length of the chord is $l = \sqrt{2dx}$; if, therefore, the value of x as determined in equation (48), be substituted instead of it, in the foregoing value of l, we shall have

$$l = \tfrac{1}{3}d\sqrt{6}. \qquad (49).$$

99. This is the proper form of the expression when adapted for numerical operation, and the practical rule which it supplies, may be expressed as follows.

RULE. *Multiply one third of the diameter of the given plane by the square root of* 6, *and the product will be the length of the chord required.* Or thus:

Multiply the given diameter by the constant number .81647, *and the product will be the length of the chord required.*

100. EXAMPLE 20. A circular plane whose diameter is equal to 36 feet, is perpendicularly immersed in a fluid, so that the upper extremity of the vertical diameter is in contact with the horizontal surface; what is the length of a chord, which being inflected from the lower extremity of the diameter, sustains a greater pressure than any other chord which can be drawn from the same point?

Here by the rule, we have

$$l = \tfrac{1}{3} \times 36 \times \sqrt{6} = 29.3929 \text{ feet},$$

and by the second part of the rule, we have

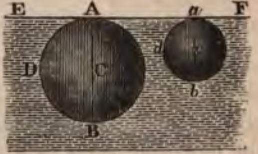
$$l = 36 \times .81647 = 29.3929 \text{ feet}.$$

PROBLEM XV.

101. If two spheres or globes of different diameters, be immersed in a fluid, in such a manner, that the uppermost point on their surface is just in contact with the horizontal surface of the fluid:—

It is required to determine the pressure on each of the spheres, and to compare the pressures with one another.

Let ABD and abd be the two spheres, whose diameters AB and ab, have their upper extremities A and a in contact with EF, the horizontal surface of the fluid.



Bisect the diameters AB and ab, respectively in the points C and c; then are the points C and c, the centres of magnitude of the respective spheres; but in a sphere, the centre of magnitude and the centre of gravity occur in the same point; therefore, C and c are the centres of gravity of the spheres ABD and abd, and AC and ac are the perpendicular depths below the surface of the fluid.

Put $D = AB$, the diameter of the greater sphere ABD,
$d = ab$, the diameter of the lesser sphere abd,
$\frac{1}{2}D = AC$, the radius of the greater sphere, or the perpendicular depth of its centre of gravity,
$\frac{1}{2}d = ac$, the radius of the lesser sphere, or the perpendicular depth of its centre of gravity,
$S =$ the surface of the greater sphere ABD,
$P =$ the pressure perpendicular to its surface,
$S' =$ the surface of the lesser sphere abd,
$p =$ the pressure perpendicular to its surface,
and $\pi = 3.1416$, the circumference of a circle whose diameter is expressed by unity.

Then, according to the principles of mensuration, the surface of a sphere or globe—

Is equal to four times the area of one of its great circles, or that whose plane passes through the centre of the sphere.

Consequently, the convex surface of the greater sphere ABD, is expressed as follows.

$$S = 3.1416 D^2,$$

and that of the lesser sphere abd, is

$$S' = 3.1416 d^2.$$

But the pressure perpendicular to any surface, is equal to the area of that surface multiplied by the perpendicular depth of the centre of gravity, and again by the specific gravity of the fluid; consequently, when the specific gravity of the fluid is denoted by unity, we have for the pressure on the surface of the greater sphere,

$$P = 3.1416 D^2 \times \tfrac{1}{2} D = 1.5708 D^3. \qquad (50).$$

and for the pressure on the lesser sphere, it is

$$p = 3.1416 d^2 \times \tfrac{1}{2} d = 1.5708 d^3;$$

hence, by comparison, we shall have

$$P : p :: D^3 : d^3.$$

Consequently, if two spheres of different diameters be placed in a fluid under similar circumstances, the pressures perpendicular to their surfaces, are to one another as the cubes of their diameters.

By the principles of mensuration, the solid content of a sphere or globe, is equal to the cube of the diameter multiplied by the constant number .5236; therefore, if c denote the solid content, we have

$$c = .5236 D^3,$$

or multiplying both sides of the equation by 3, we get

$$3c = 1.5708\,\text{D}^3;$$

consequently, by equation (50), we have

$$\text{P} = 3c;$$

or if s denote the specific gravity of the fluid, we shall obtain

$$\text{P}s = 3cs.$$

Corol. Hence we infer, that if a hollow sphere or globe be filled with an incompressible and non-elastic fluid:—

> *The whole pressure sustained by the internal surface of the sphere is equal to three times the weight of the fluid which it contains.*

102. Example 21. A hollow spherical shell or vessel, whose interior diameter is equal to 30 feet, is completely filled with water; what weight is equivalent to the pressure sustained by its internal surface?

Here, by operating according to the process indicated in equation (50), we have

$$\text{P} = 1.5708 \times 30^3 = 42411.6 \text{ cub. ft.}$$

Now, since the fluid with which the vessel is filled, is water, giving a weight of 62½ lbs. to a cubic foot, we have

$$\text{P} = 42411.6 \times 62\tfrac{1}{2} = 2650725 \text{ lbs.};$$

but 2240 lbs. are equal to one ton; therefore, the pressure on the internal surface of a hollow spherical vessel whose diameter is 30 feet, when completely filled with water, is

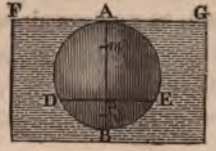
$$\text{P} = 2650725 \div 2240 = 1183\tfrac{161}{448} \text{ tons.}$$

PROBLEM XVI.

103. Suppose a sphere or globe to be immersed in an incompressible and non-elastic fluid, in such a manner, that the upper extremity of the vertical diameter is just in contact with its surface:—

> *It is required to determine through what point of the axis a horizontal plane must pass, so to divide the sphere, that the pressure on the convex surface of the lower segment, may be equal to the pressure on the convex surface of the upper.*

Let ADBE represent the sphere in question, so placed, that A the upper extremity of the vertical diameter, is just in contact with FG the surface of the fluid.



Suppose that P is the point in the vertical diameter through which the plane of division passes, separating the sphere into the segments DAE and DBE, sustaining equal pressures on their convex surfaces.

Bisect AP and BP in the points m and n; then are m and n, the points thus determined, respectively the centres of gravity of the surfaces of the spheric segments DAE and DBE,* and Am, An are their perpendicular depths below FG the horizontal surface of the fluid.

Put $D = AB$, the vertical diameter of the sphere or globe ADBE,
$d = Am$, the depth of the centre of gravity of the surface of the upper segment DAE,
$\delta = An$, the depth of the centre of gravity of the surface of the lower segment DBE,
$S =$ the surface of the upper segment,
$P =$ the pressure upon it,
$S' =$ the surface of the lower segment,
$p =$ the pressure upon it,
$s =$ the specific gravity of the fluid,
$x = AP$, the perpendicular depth of the point through which the plane of division passes, and
$\pi = 3.1416$, the circumference of the circle whose diameter is unity.

Then, according to the principles of mensuration, the convex surface of a spheric segment:—

> *Is equal to the circumference of the sphere, drawn into the versed sine or height of the segment whose surface is sought.*

And moreover, the circumference of a sphere, or the circumference of any of its great circles:—

> *Is equal to the diameter multiplied by the constant quantity* π, *or the number* 3.1416.

consequently, the convex surface of the upper segment DAE, is

$$S = D\pi x = 3.1416\,Dx,$$

and that of the lower segment DBE, is

$$S' = D\pi(D - x) = 3.1416\,D(D - x).$$

---

* It is demonstrated by the writers on mechanics, that the centre of gravity of the surface of a spheric segment, is at the middle of its versed sine or height.

But the pressure perpendicular to any surface, whatever may be its form, as we have already sufficiently demonstrated :—

> *Is equal to the area of the surface multiplied by the perpendicular depth of the centre of gravity, and again by the specific gravity of the fluid.*

Therefore, the pressure perpendicular to the convex surface of the upper segment DAE, is

$$P = 3.1416\, Dx \times \tfrac{1}{2}x \times s = 1.5708\, Dsx^2. \qquad (51).$$

and the pressure perpendicular to the convex surface of the lower segment DBE, is

$$p = 3.1416D(D-x)\times(x+\tfrac{1}{2}(D-x))\times s = 3.1416Ds\{(D-x)x+\tfrac{1}{2}(D-x)^2\} \qquad (52).$$

Now these two expressions, according to the conditions of the problem, are equal to one another; consequently, by comparison, we get

$$1.5708\, Dsx^2 = 3.1416\, Ds\{(D-x)\,x+\tfrac{1}{2}(D-x)^2\},$$

and from this, by suppressing the common quantities, we have

$$x^2 = 2\{(D-x)\,x+\tfrac{1}{2}(D-x)^2\};$$

therefore, by expanding the terms, we obtain

$$2x^2 = D^2;$$

consequently, by division and evolution, we get

$$x = \tfrac{1}{2}D\sqrt{2}. \qquad (53).$$

104. The ultimate form of this equation is extremely simple, and the practical rule which it supplies, may be expressed as follows.

> RULE. *Multiply the radius, or half the diameter of the sphere by the square root of 2, and the product will give the point in the vertical diameter through which the plane of division passes, estimated downwards from the surface of the fluid.*

105. EXAMPLE 22. A sphere or globe, whose diameter is 18 inches, is immersed in a fluid, in such a manner, that the upper extremity is just in contact with the surface; through what point of the diameter must a horizontal plane be made to pass, so to divide the sphere, that the pressures on the curve surface of the upper and lower segments may be equal to one another?

The square root of 2, is 1.4142, and half the given diameter is 9 inches; consequently, by the rule we have

$$x = 1.4142 \times 9 = 12.7278 \text{ inches.}$$

106. The preceding investigation applies to the particular case, in which the pressures on the curve surfaces of the segments are equal to one another; but in order to render the solution general, we must investigate a formula to indicate the point of division, when the pressures are to one another in any ratio whatever; for instance, that of m to n.

By expunging the common factors from the equations (51) and (52), we obtain

$$x^2 : 2\{(\mathrm{D}-x)x+\tfrac{1}{2}(\mathrm{D}-x)^2\} :: m : n;$$

therefore, by equating the products of the extreme and mean terms, we get

$$2m\{(\mathrm{D}-x)x+\tfrac{1}{2}(\mathrm{D}-x)^2\}=nx^2,$$

which, by expanding the bracketted expression, becomes

$$nx^2=m(\mathrm{D}^2-x^2),$$

or by transposition, we obtain

$$(m+n)x^2=m\mathrm{D}^2,$$

and finally, by dividing and extracting the square root, we have

$$x=\mathrm{D}\sqrt{\frac{m}{m+n}}. \qquad (54).$$

107. The general equation just investigated, is sufficiently simple in its form for every practical purpose that is likely to occur; it may therefore appear superfluous to reduce it to a rule, yet nevertheless, that nothing may be wanting for the general accommodation of our readers, we think proper to draw up the following enunciation.

> RULE. *Divide the first term of the ratio by the sum of the terms, and multiply the square root of the quotient by the diameter of the sphere; then, the product thus arising, will express the distance below the surface of the fluid, of that point through which the plane of division passes.*

It is unnecessary to propose an example for the purpose of illustrating the above rule; that which we have already given, where the values of m and n are equal to one another, being quite sufficient.

CHAPTER V.

OF THE PRESSURE OF NON-ELASTIC OR INCOMPRESSIBLE FLUIDS AGAINST THE INTERIOR SURFACES OF VESSELS HAVING THE FORMS OF TETRAHEDRONS, CYLINDERS, TRUNCATED CONES, &c.

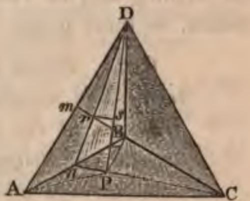
1. WHEN THE VESSEL IS IN THE FORM OF A TETRAHEDRON.

PROBLEM XVII.

108. Suppose a vessel in the form of a tetrahedron, or equilateral triangular pyramid, to be filled with an incompressible and non-elastic fluid:—

> *It is required to compare the pressure on the base with that upon the sides, and also with the weight of the fluid; the base of the vessel being parallel to the horizon.*

Let ABCD be the tetrahedron filled with fluid, of which ABC is the base parallel to the horizon, and ABD, CBD and ADC the sides or equal containing planes.



From D the vertex of the figure, let fall the perpendicular DP, upon the base or opposite side ABC; then will DP be the vertical depth of the centre of gravity of the base ABC, below the horizontal plane passing through D, the summit of the figure, or highest particle of the fluid.

Bisect AD and AB, two of the adjacent edges of the figure, in the points m and n; draw the straight lines Bm and Dn, intersecting each other in the point r; then is r the centre of gravity of the triangular plane ABD.

Through the point r draw rs perpendicular to DP, the altitude of the vessel or pyramid; then is Ds, the perpendicular depth of the centre of gravity of the triangular plane ADB, below the vertex D, or the uppermost particle of the fluid.

By the nature of the figure, the three containing planes ADB, ADC and BDC, are equal to one another, and they are also equally inclined to, or similarly situated with respect to the base ABC; consequently Ds, the perpendicular depth of the centre of gravity, is common to them all.

Now, by the property of the centre of gravity, we know, that Dr is equal to two thirds of Dn; therefore, by reason of the parallel lines nP and rs, Ds is also equal to two thirds of DP.

Put $a =$ the area of the base and each of the other containing planes,
$l =$ the length of the side of each triangular plane, or the edges of the figure,
$d =$ DP, the perpendicular depth of the centre of gravity of the base ABC,
$\delta =$ Ds, the perpendicular depth of the centre of gravity of the side ADB;
$P =$ the pressure upon the base,
$p =$ the pressure upon one of the sides,
$w =$ the weight of the fluid contained in the vessel, and
$s =$ the specific gravity.

Then, by the principles of Plane Trigonometry and the property of the right angled triangle, we have

$$\text{DP} = d = \frac{l}{2}\sqrt{4 - \sec.^2 30^\circ},$$

but by the arithmetic of sines, we know, that

$$\sec.^2 30^\circ = 1\tfrac{1}{3};$$

consequently, by substitution, we have

$$d = \frac{l}{3}\sqrt{6}.$$

Now, according to the construction of the figure and the property of the centre of gravity, it follows, that Ds is equal to two thirds of DP; hence we get

$$\text{D}s = \delta = \frac{2\,l}{9}\sqrt{6}.$$

By the nature of the figure, it is manifest, that the area of each of the triangular sides is equal to the area of the base; and by the principles of mensuration :—

> *The area of an equilateral triangle, is equal to one fourth the square of the side, multiplied by the square root of three.*

Consequently, the area of the base, and each of the containing sides of the vessel, is expressed by

$$a = \tfrac{1}{4}l^2\sqrt{3};$$

therefore, the area of the three containing equilateral triangular planes, becomes

$$3a = \tfrac{3}{4}l^2\sqrt{3};$$

hence, for the pressure upon the base, we have

$$P = \tfrac{1}{4}l^2\sqrt{3} \times \tfrac{1}{3}l\sqrt{6} \times s = \tfrac{1}{4}l^3 s\sqrt{2}, \qquad (55).$$

and the pressure upon the three containing planes, is

$$p = \tfrac{3}{4}l^2\sqrt{3} \times \tfrac{2}{9}l\sqrt{6} \times s = \tfrac{1}{2}l^3 s\sqrt{2}; \qquad (56).$$

consequently, by analogy, we shall have

$$P : p :: \tfrac{1}{4}l^3 s\sqrt{2} : \tfrac{1}{2}l^3 s\sqrt{2};$$

and this, by suppressing the common factors, becomes

$$P : p :: 1 : 2.$$

If the two equations marked (55) and (56) be added together, the sum will express the aggregate pressure upon the vessel; therefore we have

$$P + p = p' = (\tfrac{1}{4} + \tfrac{1}{2})\, l^3 s\sqrt{2} = \tfrac{3}{4}l^3 s\sqrt{2}. \qquad (57).$$

According to the principles of mensuration, the solid content of a tetrahedronal vessel, is equal to the area of its base, multiplied by one third of its perpendicular altitude; therefore, we have

$$\tfrac{1}{4}l^2\sqrt{3} \times \tfrac{1}{3}l\sqrt{6} \times \tfrac{1}{3} = \tfrac{1}{12}l^3\sqrt{2};$$

now, the weight of the contained fluid, is manifestly equal to its magnitude multiplied by the specific gravity; consequently, we obtain

$$w = \tfrac{1}{12}l^3 s\sqrt{2}; \qquad (58).$$

hence, by analogy, we get

$$P : w :: \tfrac{1}{4}l^3 s\sqrt{2} : \tfrac{1}{12}l^3 s\sqrt{2},$$

and this, by suppressing the common factors, gives

$$P : w :: 3 : 1.$$

Coról. It therefore appears, that the pressure upon the base, is to the pressure on the three sides, in the ratio of 1 to 2, and to the weight of the contained fluid in the ratio of 3 to 1; consequently, the weight of the fluid, the pressure on the base, and the pressure on the sides, are to one another as the numbers 1, 3 and 6.

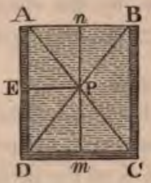
2. WHEN THE VESSEL IS IN THE FORM OF A CYLINDER.

PROBLEM XVIII.

109. If a cylindrical vessel be completely filled with an incompressible and non-elastic fluid, and so placed, that its bottom may be parallel to the horizon:—

> *It is required to compare the pressure against its bottom, with that against its upright surface, and also with the weight of the fluid which it contains.*

Let ABCD be a vertical section of a cylindrical vessel, filled with an incompressible and non-elastic fluid, whose surface AB is horizontal, and let it be required to compare the pressure exerted by the fluid on the bottom DC, with that upon the whole upright surface.



Draw the diagonals AC and BD intersecting one another in the point p, and through the point P, draw the vertical line mn, meeting DC the bottom of the vessel in the point m, and AB the surface of the fluid in n; then is m the position of the centre of gravity of the bottom, and nm its perpendicular depth below the surface AB.

Bisect AD in E, and through E draw EP parallel to AB or DC, and meeting the vertical line mn in P; then is P the position of the centre of gravity of the upright surface, and nP its perpendicular depth below AB the surface of the fluid.

Put D $=$ AB or DC, the diameter of the cylindrical vessel proposed,
$d = nm$, the perpendicular depth of the centre of gravity of the bottom DC, below AB the surface of the fluid,
$\delta = n\text{P}$, the perpendicular depth of the centre of gravity of the upright surface,
$\text{A} =$ the area of the base or bottom of the cylinder,
$\text{P} =$ the pressure upon it,
$a =$ the area of the curved or upright surface,
$p =$ the pressure upon it,
$w =$ the weight,
and $s =$ the specific gravity of the fluid.

Then, by the principles of mensuration, the area of the base or bottom of the cylindric vessel, is

$$\text{A} = .7854\,\text{D}^2,$$

and that of the upright surface, is

$$a = 3.1416Dd;$$

consequently, the pressure on the bottom becomes

$$P = .7854D^2ds, \qquad (59).$$

and for the pressure upon the upright surface, we have

$$p = 3.1416Dd\delta s; \qquad (60).$$

therefore, by analogy, we obtain

$$P : p :: .7854D^2ds : 3.1416Dd\delta s;$$

from which, by omitting the common factors, we get

$$P : p :: D : 4\delta;$$

now, by the construction of the figure, we have $\delta = \frac{1}{2}d$; therefore $4\delta = 2d$, and by substitution, we obtain

$$P : p :: D : 2d :: \tfrac{1}{2}D : d.$$

Hence it appears, that the pressure upon the bottom of a cylindrical vessel, is to the pressure upon its upright surface, as the radius of the base is to the perpendicular altitude.

Since the entire pressure sustained by a cylindrical vessel, is equal to the sum of the pressures on the bottom and the upright sides, it follows, that

$$P + p = p' = .7854(D + 4\delta)Dds,$$

or substituting $\frac{1}{2}d$ for δ, we shall obtain

$$p' = .7854(D + 2d)Dds. \qquad (61).$$

It is demonstrated by the writers on mensuration, that the solid content of a cylinder, is equal to the area of its base, drawn into its perpendicular altitude; therefore, we have

$$C = .7854D^2d,$$

where C denotes the solid content of the cylinder.

Now, it is manifest, that the weight of an incompressible and non-elastic fluid, is equal to its magnitude drawn into its specific gravity; hence we have

$$w = .7854D^2ds;$$

but this is precisely the expression which we have given in equation (59), for the pressure perpendicular to the bottom of the vessel; consequently, the weight of the fluid, and the pressure on the bottom of the vessel, are equal to one another; hence, the following inference.

110. When the sides of a vessel of any form whatever, are perpendicular, and its base parallel to the horizon :—

The pressure perpendicular to the base of the vessel, is equal to the whole weight of the fluid which it contains.

This is manifest, for the whole pressure of the fluid is sustained by the base and the sides together, and the sides being in the direction of gravity, sustain no part of the pressure which is exerted perpendicularly downwards; consequently, the whole weight of the fluid is sustained by the base.

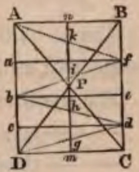
3. WHEN THE PRESSURE UPON THE ANNULI OF A CYLINDER IS TO BE DETERMINED.

PROBLEM XIX.

111. If a cylindrical vessel whose bottom is parallel, and sides perpendicular to the horizon, be filled with an incompressible and non-elastic fluid:—

> *It is required to divide the concave surface, into any number n of horizontal annuli, in such a manner, that the pressure on each annulus shall be equal to the pressure on the bottom of the vessel.*

Let ABCD be a vertical section, passing along the axis of the cylinder, or vessel containing the fluid, whose surface is AB; draw the diagonals AC and BD intersecting one another in the point P; then is P the centre of gravity of the cylindrical surface.



Through P the point of intersection, draw the vertical line mn parallel to AD or BC, and let a, b and c be the points, which with the extremities A and D of the side AD, terminate the several annuli: then, through the points a, b and c, and parallel to AB or DC, draw the straight lines af, be and cd, cutting BC the opposite side of the section in the points f, e and d.

Draw the zigzag diagonals Af, fb, bd and dD, intersecting the vertical line mn in the points k, i, h and g; then are the points thus determined, respectively the centres of gravity of the several annuli, into which the concave surface of the vessel is supposed to be divided; and nk, ni, nh and ng, are the respective depths below the surface of the fluid, nP being the depth of the centre of gravity of the whole upright surface of the cylindrical vessel, and nm the vertical depth of the bottom.

Put $\mathrm{D} = \mathrm{AB}$ or DC, the diameter of the proposed cylindrical vessel,
$\mathrm{A} =$ the area of its bottom,
$d = nm$, the whole perpendicular depth, or altitude of the cylinder,
$x = \mathrm{A}a$, the breadth of the first annulus,
$x' = ab$, the breadth of the second,
$x'' = bc$, the breadth of the third,
$x''' = c\mathrm{D}$, the breadth of the fourth, and so on, to any number of annuli n,
$\mathrm{P} =$ the pressure on the concave surface of the cylinder, or the sum of the pressures on the several annuli into which it is divided,
$p =$ the pressure on the bottom of the vessel, and each of the several annuli,
$\pi = 3.1416$ the circumference of a circle whose diameter is unity, and
$s =$ the specific gravity of the fluid.

Then we have, $nk = \frac{1}{2}x$; $ni = x + \frac{1}{2}x'$; $nh = x + x' + \frac{1}{2}x''$, and $ng = x + x' + x'' + \frac{1}{2}x'''$; and by the principles of mensuration, the area of the bottom of the vessel, is

$$\mathrm{A} = \tfrac{1}{4}\pi\,\mathrm{D}^2,$$

and by Proposition (1), the pressure upon it, is

$$p = \tfrac{1}{4}\pi\,\mathrm{D}^2 ds = .7854\,\mathrm{D}^2 ds, \qquad (62).$$

Again, by the principles of mensuration, the concave surfaces of the respective annuli are as follows, viz.

For the first annulus, we have $\pi\,\mathrm{D}\,x = 3.1416\,\mathrm{D}\,x$, the surface,
——— second ——————— $\pi\,\mathrm{D}\,x' = 3.1416\,\mathrm{D}\,x'$, ———
——— third ——————— $\pi\,\mathrm{D}\,x'' = 3.1416\,\mathrm{D}\,x''$, ———
——— fourth ——————— $\pi\,\mathrm{D}\,x''' = 3.1416\,\mathrm{D}\,x'''$, ———
——— &c. ——————— &c. = &c.

And by Proposition (1), the pressures perpendicular to these surfaces, are respectively as below, viz.

For the first annulus, the pressure is $p = 1.5708\,\mathrm{D}x^2 s$,
——— second ——————— $p = 3.1416\,\mathrm{D}x'\,s(x + \frac{1}{2}x')$,
——— third ——————— $p = 3.1416\,\mathrm{D}x''\,s(x + x' + \frac{1}{2}x'')$,
——— fourth ——————— $p = 3.1416\,\mathrm{D}x'''\,s(x + x' + x'' + \frac{1}{2}x''')$,
——— &c. ——————— &c. &c.

Now each of these pressures, according to the conditions of the problem, is equal to the pressure upon the bottom, exhibited in the equation (62); consequently, by comparison, we have

$$1.5708\,\mathrm{D}x^2s = .7854\,\mathrm{D}^2ds,$$

and casting out the common factors, we get

$$2x^2 = \mathrm{D}d\,;$$

therefore, by division and evolution, we have

$$x = \tfrac{1}{2}\sqrt{2\mathrm{D}d}. \qquad (63).$$

By proceeding in a similar manner for the breadth of the second annulus, we shall obtain

$$3.1416\,\mathrm{D}x's\,(x + \tfrac{1}{2}x') = .7854\,\mathrm{D}^2ds,$$

and this, by expunging the common terms, becomes

$$4x'\,(x + \tfrac{1}{2}x') = \mathrm{D}d\,;$$

therefore, by substituting for x, its value as expressed in equation (63), we shall get

$$x'^2 + \sqrt{2\mathrm{D}d}.x' = \tfrac{1}{2}\mathrm{D}d\,;$$

complete the square, and we have

$$x'^2 + \sqrt{2\mathrm{D}d}x' + \tfrac{1}{2}\mathrm{D}d = \mathrm{D}d\,;$$

and finally, extracting the square root and transposing, we obtain

$$x' = \tfrac{1}{2}(2 - \sqrt{2})\,\sqrt{\mathrm{D}d}. \qquad (64).$$

Again, by performing a similar process for the breadth of the third annulus, we shall have

$$3.1416\,\mathrm{D}x''s\,(x + x' + \tfrac{1}{2}x'') = .7854\,\mathrm{D}^2ds,$$

from which, by casting out the common quantities, we get

$$4x''\,(x + x' + \tfrac{1}{2}x'') = \mathrm{D}d\,;$$

therefore, by substituting for x and x', their values as expressed in the equations marked (63) and (64), and we shall obtain

$$4x''\,\{\tfrac{1}{2}\sqrt{2\mathrm{D}d} + \tfrac{1}{2}(2 - \sqrt{2})\,\sqrt{\mathrm{D}d} + \tfrac{1}{2}x''\} = \mathrm{D}d,$$

and this, by a little reduction and proper arrangement, gives

$$x''^2 + 2\,\sqrt{\mathrm{D}d}\,.\,x'' = \tfrac{1}{2}\mathrm{D}d\,;$$

complete the square, and we obtain

$$x''^2 + 2\,\sqrt{\mathrm{D}d}\,.\,x'' + \mathrm{D}d = \tfrac{1}{2}\mathrm{D}d\,;$$

consequently, by extracting the square root and transposing, we get

$$x'' = \tfrac{1}{2}\sqrt{6} - 2)\sqrt{\mathrm{D}d}. \qquad (65).$$

Pursuing a similar train of reasoning for the breadth of the fourth annulus, we shall obtain

$$3.1416\,\mathrm{D}x'''s\,(x + x' + x'' + \tfrac{1}{2}x''') = .7854\,\mathrm{D}^2ds,$$

and by suppressing the common factors, we have

$$4x'''\,(x + x' + x'' + \tfrac{1}{2}x''') = \mathrm{D}d\,;$$

sustitute in this equation, the values of x, x' and x'' as represented in the equations marked (63), (64), and (65), and we get

$$2x'''\{\sqrt{2\text{D}d}+(2-\sqrt{2})\sqrt{\text{D}d}+(\sqrt{6}-2)\sqrt{\text{D}d}+x'''\}=\text{D}d,$$

and this, by a little further reduction, becomes

$$x'''^2+\sqrt{6\text{D}d}\,.\,x'''=\tfrac{1}{2}\text{D}d;$$

therefore, by completing the square, we obtain

$$x'''^2+\sqrt{6\text{D}d}\,.\,x''+\tfrac{3}{2}\text{D}d=2\text{D}d;$$

and finally, by extracting the square root and transposing, we have

$$x'''=\tfrac{1}{2}(2\sqrt{2}-\sqrt{6})\sqrt{\text{D}d}. \qquad (66).$$

112. And thus we may proceed to any extent at pleasure; that is, to any number of annuli within the limit of possibility; for it is manifest, from the nature of the problem, that impossible cases may be proposed, but the limit can easily be ascertained in the following manner.

It is obvious, that the sum of the breadths of the several annuli, is equal to the whole depth of the vessel; and that the sum of the pressures is equal to the pressure on the concave surface; but in the problem immediately preceding, we have demonstrated that the pressure on the bottom of a cylindrical vessel, is to that upon its upright surface, as the radius of the base is to the perpendicular altitude.

Now, according to the conditions of the question, the pressure on each annulus is equal to that upon the base; consequently, in order that the problem may be possible, the depth of the vessel must be equal to the radius of the base, drawn into the number of annuli.

If instead of D the diameter of the cylindrical vessel, we substitute 2R its equivalent in terms of the radius, the preceding equations (63), (64), (65), and (66), will become transformed into

$$x=(\sqrt{1}-\sqrt{0})\sqrt{\text{R}d}, \qquad (67).$$

$$x'=(\sqrt{2}-\sqrt{1})\sqrt{\text{R}d}, \qquad (68).$$

$$x''=(\sqrt{3}-\sqrt{2})\sqrt{\text{R}d}, \qquad (69).$$

$$x'''=(\sqrt{4}-\sqrt{3})\sqrt{\text{R}d}, \qquad (70).$$

From these equations the law of induction becomes manifest, and the general expression for the breadth of the n^{th} annulus, is

$$x^{n}=(\sqrt{n}-\sqrt{n-1})\sqrt{\text{R}d}. \qquad (71).$$

113. And from the above general form of the equation, the following practical rule may be derived, for calculating the breadth of any proposed annulus, independently of the breadths of those which precede it.

> **Rule.** *From the square root of the number which expresses the place of the required annulus, subtract the square root of that number minus unity; then, multiply the remainder by the geometric mean between the altitude of the vessel and the radius of its base, and the product will give the breadth of the required annulus.*

114. **Example.** A cylindrical vessel has the radius of its base, and its perpendicular depth, respectively equal to 4 and 24 feet; now, supposing the concave surface to be divided into 6 horizontal annuli, such, that the pressure upon each shall be equal to the pressure upon the base; required the breadth of the fourth annulus?

By performing the operation as directed in the preceding rule, we shall obtain

$$x''' = (\sqrt{4} - \sqrt{3}) \times \sqrt{4 \times 24} = 2.625 \text{ feet nearly.}$$

The annulus which we have just determined, corresponds to the fourth of the preceding class of equations, or that marked (69), and the distance of its centre of gravity below the surface of the fluid, or its position with respect to the bottom or top of the vessel, can easily be ascertained.

The area of the cylinder's base, is

$$A = 3.1416\,R^2;$$

the pressure which it sustains, is

$$p = 3.1416\,R^2 d = 1206.3744,$$

and this is equal to the pressure on the annulus.

Now, according to the writers on mensuration, the area of the annulus, or the curved surface of a cylinder, whose radius is 4 feet and its perpendicular altitude 2.625 feet, is expressed as follows, viz.

$$6.2832 \times 4 \times 2.625 = 65.9736.$$

If therefore, we divide the pressure on the base of the vessel, by the area of the annulus, the depth of its centre of gravity will become known; thus, we have

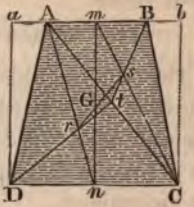
$$\delta = \frac{1206.3744}{65.9736} = 18.28 \text{ feet nearly.}$$

4. WHEN THE VESSEL ASSUMES THE FORM OF A TRUNCATED CONE, THE BASE OF WHICH IS ALSO THE BOTTOM OF THE VESSEL, AND ITS AXIS PERPENDICULAR TO THE HORIZON.

PROBLEM XX.

115. If a vessel in the form of the frustum of a cone, be filled with an incompressible and non-elastic fluid, and have its axis perpendicular to the horizon:—

> *It is required to compare the pressure on the bottom of the vessel with that upon its curved surface, and also with the weight of the fluid which it contains, both when the sides of the vessel converge, and when they diverge from the extremities of the bottom.*



Let ABCD represent a vertical section of a vessel in the form of the frustum of a cone, and filled with an incompressible and non-elastic fluid whose horizontal surface is AB; produce AB both ways, to any convenient distance, and through D and C the extremities of the bottom diameter, draw Da and Cb respectively perpendicular to DC, and meeting AB produced in the points a and b; then is abCD the vertical section, passing along the axis of the cylinder which circumscribes the conic frustum.

Bisect AB and DC respectively in the points m and n, and draw the straight line mn; then, because the figure ABCD is symmetrical with respect to the axis mn, it follows, that mn bisects the figure or trapezoid ABCD, and consequently passes through its centre of gravity.

Draw the diagonal AC, dividing the figure ABCD into the two triangles ABC and ADC; then it is manifest, that the common centre of gravity of the two triangles, and that of the trapezoid constituted by their sum, must occur in one and the same point; therefore, bisect the diagonal AC in the point t, and draw An and Dt intersecting each other in the point r, and Cm, Bt intersecting in s; then are r and s the centres of gravity of the triangles ADC and ABC; draw rs intersecting mn in G, and G will be the centre of gravity of the trapezoid ABCD.

Now, it is demonstrated by the writers on mechanics, that the centre of gravity of the surface of a conic frustum:

Is situated in the axis, and at the same distance from its extremities, as is the centre of gravity of the trapezoid, which is a vertical section passing along the axis of the solid.

Therefore, since by the construction, the point G has been shown to be the centre of gravity of the trapezoid, it is also the centre of gravity of the surface of the conic frustum, and mG is its perpendicular depth below the surface of the fluid.

Put $A =$ the area of the base or bottom of the vessel, whose diameter is DC,
$D = mn$, the perpendicular depth of its centre of gravity, or the length of the axis of the vessel,
$a =$ the curve surface of the conic frustum,
$d = mG$, the perpendicular depth of the centre of gravity,
$\beta =$ DC, the diameter of the base or bottom of the vessel,
$\delta =$ AB, the diameter of the top,
$P =$ the pressure on the bottom,
$p =$ the pressure on the curve surface,
$w =$ the weight, and s the specific gravity of the fluid.

Then, according to the principles of mensuration, the area of the lower base of the conic frustum, or the bottom of the vessel on which the fluid presses, becomes

$$A = .7854\beta^2,$$

and consequently, the pressure which it sustains, is

$$P = .7854\beta^2 D s. \qquad (72).$$

In the next place, the area of the curved surface of the conic frustum, or the sides of the vessel containing the fluid, is

$$a = 1.5708\,(\beta + \delta) \times \sqrt{D^2 + \tfrac{1}{4}(\beta - \delta)^2}\,;$$

and therefore, the pressure which it sustains, is

$$p = 1.5708\,(\beta + \delta)\, ds\, \sqrt{D^2 + \tfrac{1}{4}(\beta - \delta)^2}. \qquad (73).$$

Now, according to the writers on mechanics, the depth of the centre of gravity of the trapezoid ABCD, below the horizontal line AB, is obtained in the following manner:

$$3\,(\beta + \delta) : D :: 2\beta + \delta : d,$$

and by equating the products of the extremes and means, we get

$$3d\,(\beta + \delta) = D\,(2\beta + \delta),$$

therefore, dividing by $3\,(\beta + \delta)$, we obtain

$$d = \frac{D\,(2\beta + \delta)}{3(\beta + \delta)}. \qquad (74).$$

Let this value of d be substituted instead of it, in the equation marked (74), and we shall have for the pressure on the curved surface of the vessel

$$p = .5236\,\mathrm{D}s\,(2\beta + \delta)\,\sqrt{\mathrm{D}^2 + \tfrac{1}{4}(\beta - \delta)^2}\,; \qquad (75).$$

consequently, by comparing the equations (72) and (74), we get

$$\mathrm{P} : p :: .7854\beta^2\mathrm{D}s : .5236\,\mathrm{D}s\,(2\beta + \delta)\,\sqrt{\mathrm{D}^2 + \tfrac{1}{4}(\beta - \delta)^2},$$

and this, by suppressing the common factors, becomes

$$\mathrm{P} : p :: 3\beta^2 : 2\,(2\beta + \delta)\,\sqrt{\mathrm{D}^2 + \tfrac{1}{4}(\beta - \delta)^2}. \qquad (76).$$

If δ, the upper diameter of the frustum vanishes, the figure becomes a complete cone, and consequently, the pressure upon the base, is to that upon the curve surface, as three times the diameter of the cone, is to four times its slant height; that is

$$\mathrm{P} : p :: 3\beta : 4\sqrt{\mathrm{D}^2 + \tfrac{1}{4}\beta^2}. \qquad (77).$$

According to the principles of solid geometry, the capacity of the conic frustum, or the quantity of fluid which the vessel contains, is

$$\mathrm{C} = .2618\,\mathrm{D}\,(\beta^2 + \beta\delta + \delta^2),$$

where C denotes the solid content of the vessel.

But the weight of any quantity or mass of fluid, varies directly as the magnitude and specific gravity conjointly; consequently, the weight of fluid in the vessel, is expressed by

$$w = .2618\,\mathrm{D}s\,(\beta^2 + \beta\delta + \delta^2). \qquad (78).$$

Hence, if the equations marked (72) and (78), be compared with each other, we shall obtain

$$\mathrm{P} : w :: 3\beta^2 : (\beta^2 + \beta\delta + \delta^2), \qquad (79).$$

and when δ vanishes, the vessel becomes a complete cone, and consequently, we get

$$\mathrm{P} : w :: 3 : 1, \qquad (80).$$

It therefore appears, that the pressure against the bottom of a conical vessel, when filled with an incompressible and non-elastic fluid, (the bottom being downwards):

> *Is equivalent to three times the weight of the fluid which it contains.*

The solidity of the cylinder circumscribing the conic frustum, of which $ab\mathrm{CD}$ is a vertical section, is

$$\mathrm{C}' = .7854\,\beta^2\mathrm{D},$$

where C′ denotes the capacity of the cylinder circumscribing the vessel;

and because the weight of any quantity or mass of fluid, is proportional to the magnitude and specific gravity jointly; it follows, that if w' denote the weight of the circumscribing column of fluid, we obtain

$$w' = .7854\,\beta^2 \mathrm{D}s. \qquad (81).$$

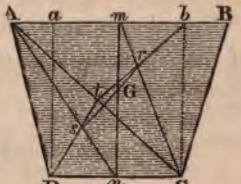
Corol. Now this expression is precisely the same, as that which we obtained for the pressure on the bottom, indicated by the equation marked (72); hence it appears, that when the sides of the vessel converge from the extremities of the diameter of its base towards each other :—

> *The pressure on the base or bottom of the vessel, is equal to the weight of a column of the fluid, of the same magnitude as the cylinder circumscribing the conic frustum, or the vessel by which the fluid is contained.*

But the circumscribing cylinder is manifestly greater than the conic frustum; consequently, the pressure upon the base or bottom of the vessel, is greater than the weight of the fluid which it contains; and it is obvious, that the additional pressure arises from the re-action of the converging sides.

5. WHEN THE VESSEL REPRESENTS AN INVERTED TRUNCATED CONE, WITH ITS AXIS PERPENDICULAR TO THE HORIZON.

116. If the sides of the vessel diverge from the extremities of the base, as represented in the subjoined diagram; then, it may be shown, that the weight of the fluid which the vessel contains, exceeds the pressure upon its base.



Let ABCD be a vertical section, passing along the axis of a vessel in the form of a conic frustum, and which is filled with an incompressible fluid whose horizontal surface is AB; the greater base of the frustum being uppermost, or which is the same thing, the sides diverging from the extremities of the lower diameter.

Bisect the diameters AB and CD respectively in the points m and n; draw mn, and through the points D and C, the extremities of DC, draw the straight lines Da and Cb respectively parallel to mn, and meeting AB in the points a and b; then is abCD a vertical section passing along the axis of the inscribed cylinder.

Draw the diagonal AC, dividing the trapezoid ABCD, into the two triangles ABC and ADC; bisect the diagonal AC in the point t, and draw $\mathrm{B}t$ and $\mathrm{D}t$, which will be intersected by the straight lines $\mathrm{C}m$ and $\mathrm{A}n$ in the points r and s; then are r and s respectively the centres of gravity of the triangles ABC and ADC.

Join the points r and s, by the straight line rs, intersecting mn in the point G; then it is obvious, that the common centre of gravity of the triangles ABC and ADC, (which coincides with that of the trapezoid ABCD), must occur in the line rs, which joins their respective centres.

Now, because the trapezoid ABCD is symmetrically situated with respect to the axis mn, it follows, that its centre of gravity must occur in that line; but we have shown above, that it also occurs in the line rs, it consequently must be situated in the point G, where these lines intersect one another; hence, the centre of gravity of the surface of the conic frustum occurs at the point G, and $m\mathrm{G}$ is its perpendicular depth below AB the upper surface of the fluid.

Put $\mathrm{A} =$ the area of the lower base or bottom of the vessel, whose diameter is DC,

$\mathrm{P} =$ the pressure perpendicular to its surface, or the weight of a quantity of fluid equal to the inscribed cylinder,

$\mathrm{D} = mn$, the axis of the frustum, or the perpendicular depth of the centre of gravity of the bottom,

$a =$ the area of the curve surface of the vessel or conic frustum,

$p =$ the pressure perpendicular to the curve surface,

$d = m\mathrm{G}$, the perpendicular depth of its centre of gravity,

$\delta = \mathrm{DC}$, the diameter of the lower base or bottom of the vessel,

$\beta = \mathrm{AB}$, the diameter of the top or upper base,

$w =$ the weight, and s the specific gravity of the contained fluid.

Then, by the principles of mensuration, the area of the lower base of the conic frustum, or the bottom of the vessel on which the fluid presses, is

$$\mathrm{A} = .7854\,\delta^2,$$

and consequently, the pressure upon it, is

$$\mathrm{P} = .7854\,\delta^2 \mathrm{D} s. \qquad (82).$$

This equation, having δ^2 instead of β^2, is the same as that which we obtained for the pressure on the bottom in the preceding case, when the greater base of the vessel was downwards; it therefore follows, since our notation is adapted to the same parts of the vessel, that notwithstanding the inversion, the pressure on the curved surface of the conic frustum, will still be expressed as in the equation marked (73); consequently, we have

$$\mathrm{P} : p :: .7854\,\delta^2 \mathrm{D} s :: 1.5708\,(\beta + \delta)\, d s \sqrt{\mathrm{D}' + \tfrac{1}{4}(\beta - \delta)^2},$$

or by expunging the common quantities, we get

$$P : p :: \delta^2 D : 2(\beta + \delta)\, d \sqrt{D^2 + \tfrac{1}{4}(\beta - \delta)^2}.$$

But the writers on mechanics have demonstrated, that the depth of the centre of gravity of the trapezoid ABCD, below the horizontal line AB, is expressed as follows, viz.

$$d = \frac{D(\beta + 2\delta)}{3(\beta + \delta)}; \qquad (83).$$

let therefore, this value of d be substituted instead of it in the above analogy, and we shall obtain

$$P : p :: 3\delta^2 : 2(\beta + 2\delta) \sqrt{D^2 + \tfrac{1}{4}(\beta - \delta)^2}.$$

If δ, the diameter of the bottom or lower base should vanish, the vessel becomes a complete cone with its vertex downwards, in which case, the value of d as expressed in the equation marked (83), is

$$d = \frac{D(\beta + 2 \times 0)}{3(\beta + 0)} = \tfrac{1}{3} D.$$

Let this value of d be substituted instead of it, in the equation marked (73), and suppose δ to vanish; then, the pressure on the concave surface of a conical vessel with its vertex downwards, becomes

$$p = .5236\, \beta D s \sqrt{D^2 + \tfrac{1}{4}\beta^2}. \qquad (84).$$

The solid content of the inscribed cylinder, of which the vertical section passing along the axis is abcD, becomes

$$c' = .7854\, \delta^2 D,$$

and as we have already stated, its weight is proportioned to its magnitude drawn into the specific gravity; hence we have

$$w' = .7854\, \delta^2 D s;$$

but this is the same expression which indicates the pressure on the bottom, as exhibited in the equation marked (82); hence it follows, that the pressure on the bottom or lower base of the conic frustum, when the sides diverge from the extremities of its diameter,

Is equal to the weight of a column of the fluid, of the same magnitude as the cylinder inscribed in the conic frustum.

But the solid content of the inscribed cylinder, and consequently its weight, is manifestly less than the content of the vessel; hence we infer, that when the sides of the vessel diverge from the extremities of the diameter of its bottom, the pressure on the bottom is less than the weight of the fluid which it contains, the remaining weight being supported by the resistance of the diverging sides.

117. In order to compare the pressure on the bottom of the vessel, with the weight of the fluid which it contains, we must again have recourse to the principles of solid geometry; from which we learn, that the solid content of a conic frustum, whose diameters are denoted by β and δ, and its perpendicular altitude by D, is

$$\mathrm{c} = .2618\,\mathrm{D}\,(\beta^2 + \beta\delta + \delta^2),$$

and consequently, its weight becomes

$$w = .2618\,\mathrm{D}s\,(\beta^2 + \beta\delta + \delta^2); \qquad (85).$$

therefore, by comparing this equation with that marked (82), we get

$$\mathrm{P} : w :: 3\delta^2 : (\beta^2 + \beta\delta + \delta^2). \qquad (86).$$

Again, if we compare the equations marked (73 and (85) with one another, we shall have

$$p : w :: 1.5708\,(\beta + \delta)\,ds\sqrt{\mathrm{D}^2 + \tfrac{1}{4}(\beta - \delta)} : .2618\,\mathrm{D}s\,(\beta^2 + \beta\delta + \delta^2);$$

if therefore, we expunge the common quantities, from the third and fourth terms of the above analogy, and in the third term, substitute the value of d as it is expressed in the equation (83), then we shall obtain

$$p : w :: 2\,(\beta + 2\delta)\,\sqrt{\mathrm{D}^2 + \tfrac{1}{4}(\beta - \delta)^2} : (\beta^2 + \beta\delta + \delta^2).$$

When δ vanishes, or when the vessel becomes a complete cone with its vertex downwards, the preceding analogy gives

$$p : w :: 2\,\sqrt{\mathrm{D}^2 + \tfrac{1}{4}\beta^2} : \beta.$$

In complying with the conditions of the 20th problem, the foregoing investigation has been conducted on the supposition, that the vessel in question is in the form of the frustum of a cone; but the attentive reader will readily perceive, that the same mode of procedure will apply to the frustum of any other regular pyramid, and the resulting formulæ will partake of similar forms and combinations, differing only in so far as depends upon the constant numbers which express their respective areas and solidities; it is therefore unnecessary to pursue the inquiry further, taking it for granted, that by a careful perusal of what has been done above, no difficulty will be met with in applying the same principles to any other case of form or condition that is likely to occur.

Corol. 1. By the preceding investigation, then, and the formulæ arising from it, we learn, that by causing the sides of a vessel, which is filled with an incompressible and non-elastic fluid, to converge or diverge from the extremities of the base, supposed to be horizontal:—

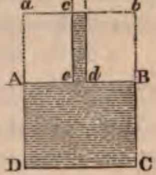
> *The pressure on the base, may be greater or less than the weight of the fluid which the vessel contains, in any proportion whatever.*

2. Upon these principles therefore, and others of a similar nature, which we have mentioned at the outset, is explained the paradoxical property of non-elastic fluids :—

> *That the pressure on the bottom of a vessel filled with fluid, does not depend upon its quantity, but solely upon the perpendicular altitude of its highest particles above the bottom of the vessel or the surface by which the pressure is sustained.*

3. And from the property here propounded, is deduced the remarkable and important principle :—

> *That any quantity of fluid however small, may be made to balance, or hold in equilibrio, any other quantity, however great.*



Let the upright or vertical section of a vessel containing an incompressible and non-elastic fluid, be such as is represented by ABCD in the annexed diagram, and let *cd* be the corresponding section of a small pipe or tube inserted into its upper surface at the point *d*.

Then, supposing the vessel and the tube to be filled with fluid as far as the point *c*; it is manifest from the first case of the preceding problem, that the pressure upon DC the bottom of the vessel, is precisely the same as if it were entirely filled to the height *acb*; for the pressure upon the bottom DC, is equal to the weight of a fluid column, the diameter of whose base is DC and its perpendicular altitude *a*D or *b*C; but this is evidently greater than the weight of the fluid in the vessel, and by increasing the height of fluid in the tube, the pressure on the bottom will be increased in the same proportion, while the actual increase of weight is very small, being only in proportion to the increase of pressure, as the area of a section of the tube is to the area of the bottom.

PROBLEM XXI.

118. Having given the diameter and perpendicular height of a cylindrical vessel, together with the diameter of a tube fixed vertically into the top of it :—

It is required to find the length of the tube, such, that when it and the vessel are filled with an incompressible fluid, the pressure on the bottom of the vessel may be equal to any number of times the fluid's weight.

In resolving this problem, it will be sufficient to refer to the preceding diagram, because a separate construction would exhibit no variety; for this purpose then,

Put $D = DC$, the diameter of the base of the cylindrical vessel,
$A =$ the area of its base,
$h = AD$, the altitude, or perpendicular depth of the vessel,
$C =$ the capacity, or solid content,
$P =$ the pressure on the bottom,
$d = ed$, the diameter of the tube inserted in the top of the vessel,
$a =$ the area of its horizontal section,
$c =$ the capacity, or solid content of the tube,
$w =$ the weight of the fluid in the vessel, and w' the weight of that in the tube;
$s =$ the specific gravity of the fluid,
$n =$ the number of times the pressure exceeds the weight,
and $x =$ the required length of the tube.

Then, according to the principles of mensuration, the area of the bottom of the vessel becomes

$$A = .7854 D^2,$$

and that of a horizontal section of a tube, is

$$a = .7854 d^2;$$

and again, by the geometry of solids, the capacity of the vessel is

$$C = .7854 D^2 h,$$

and for the capacity of the tube, we have

$$c = .7854 d^2 x;$$

the respective weights being

$$w = .7854 D^2 hs, \text{ and } w' = .7854 d^2 xs;$$

consequently, the whole weight of the fluid in the vessel and tube, is

$$w + w' = .7854 s (D^2 h + d^2 x).$$

Now, we have shown above, that the pressure on the bottom of the vessel is equal to the weight of a fluid cylinder, whose diameter is DC and altitude aD; consequently, the pressure on the bottom is expressed by

$$P = .7854 D^2 s (h + x);$$

and this, according to the conditions of the problem, is equal to n times the entire weight; hence we have

$$.7854\,\text{D}^2 s\,(h+x) = .7854\,n s\,(\text{D}^2 h + d^2 x),$$

therefore, by casting out the common factors, we get

$$\text{D}^2 (h+x) = n\,(\text{D}^2 h + d^2 x);$$

or by separating the terms and transposing, we get

$$(\text{D}^2 - n d^2)\,x = \text{D}^2 h\,(n-1),$$

from which, by division, we obtain

$$x = \frac{\text{D}^2 h\,(n-1)}{\text{D}^2 - n d^2}. \qquad (87).$$

119. It may be perhaps proper to illustrate this case by an example; but in the first place, it becomes necessary to give the rule by which the operation is to be performed.

> RULE. *From the number of times which the pressure on the bottom of the vessel, is proposed to exceed the weight of the fluid, subtract unity; multiply the remainder by the square of the vessel's diameter, drawn into its depth or perpendicular altitude, and the result will be the dividend.*
>
> *Then, from the square of the vessel's diameter, subtract n times the square of the diameter of the tube, and divide the above dividend by the remainder for the length of the tube required.*

120. EXAMPLE. If the perpendicular height of a cylindric vessel be 18 inches, its diameter 5 inches; the diameter of a tube fixed to the top of the vessel one inch; and if the vessel and tube be filled with an incompressible and non-elastic fluid, till the pressure on the bottom of the vessel is equal to twelve times the entire weight of the fluid; what is the length of the tube into which the fluid is poured?

Here, by proceeding according to the rule, it is

$$n - 1 = 12 - 1 = 11,$$

$$\text{D}^2 h = 5 \times 5 \times 18 = 450;$$

therefore, by multiplication, we obtain

$$\text{D}^2 h\,(n-1) = 450 \times 11 = 4950 \text{ dividend.}$$

Again, to determine the divisor, we have

$$\text{D}^2 - n d^2 = 5 \times 5 - 12 = 13;$$

consequently, by division, we obtain

$$x = 4950 \div 13 = 380\tfrac{10}{13} \text{ inches.}$$

Corol. 1. If the tube, instead of being fixed perpendicularly to the top of the vessel, were inserted obliquely into any part of its sides and inclined upwards, the principle above exemplified would still obtain; and the pressure in the narrow tube may be produced, not merely by the addition of a little fluid, but by the application of any kind of force, such as the working of a piston and the like.

2. If the bottom, or the cover of the cylindric vessel be made moveable, the pressure on either may be brought to bear on any one point of an external body, and may then produce an inconceivable compression, as is very successfully done in the *Hydrostatic Press*, an instrument, which, on account of the simplicity of its application, its expeditious performance, and the almost unlimited extent of its power, is altogether without a parallel in the annals of mechanical invention, and the numerous purposes to which it is applied, entitle it to no small share of popular approbation.

This machine is not only used for pressing bodies together, with a view of diminishing their bulk, in order to render them the more easily stowable; but it is equally applicable to the operation of drawing and lifting great loads, and overcoming immense resistances, however opposed to its action; even piles, which have been driven to a great depth for the purpose of forming coffer-dams, can be drawn by it with the greatest facility, and moreover, trees of the greatest size and most tenacious growth, offer but a feeble resistance to its energy; and in addition, iron bolts and cables, capable of holding the largest ships in the British navy, are totally incompetent to resist its influence.

An instrument possessing such immense power in combination with so many other advantages, such as cheapness of construction, portability, and simplicity of application, certainly merits the greatest attention, and too many attempts cannot be made to simplify the theory, and render its operations easily understood; we shall therefore, in the following pages, endeavour to unfold the principles, and to describe its construction and mode of operation.

CHAPTER VI.

THE THEORY OF CONSTRUCTION AND SCIENTIFIC DESCRIPTION OF SOME HYDROSTATIC ENGINES.

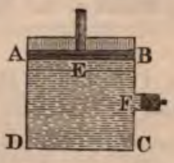
1. OF THE HYDROSTATIC OR BRAMAH PRESS.

PROPOSITION II.

121. If there be any number of pistons of different magnitudes, any how applied to apertures in a cylindrical vessel filled with an incompressible and non-elastic fluid :—

> *The forces acting on the piston to maintain an equilibrium, will be to one another as the areas of the respective apertures, or the squares of the diameters of the pistons.*

Let ABCD represent a section passing along the axis of a cylindrical vessel filled with an incompressible and non-elastic fluid, and let E, F be two pistons of different magnitudes, connected with the cylinder and closely fitted to their respective apertures or orifices; the piston F being applied to the aperture in the side of the vessel, and the piston E occupying an entire section of the cylinder or vessel, by which the fluid is contained.



Then, because by the nature of fluidity, the pressures on every part of the pistons E and F, are mutually transmitted to each other through the medium of the intervening fluid; it follows, that these pressures will be in a state of equilibrium when they are equal among themselves.

Now, it is manifest, that the sum of the pressures propagated by the piston E, is proportional to the area of a transverse section of the cylinder; and in like manner, the sum of the pressures propagated by the piston F, is proportional to the area of the aperture which it occupies; consequently, an equilibrium must obtain between these pressures :—

When the forces on the pistons, are to one another, respectively as the areas of the apertures or spaces which they occupy.

And it is obvious, that the same thing will take place, whatever may be the number of the pistons pressed.

Hence it appears, that by taking the areas of the pistons E and F, in a proper ratio to one another, we can, by means of an incompressible fluid, produce an enormous compression, and that too by the application of a very small force.

Put $P =$ the force or pressure on the piston E,
$A =$ the area of the orifice which it occupies,
$p =$ the pressure on the piston F, and
$a =$ the area of the orifice or space to which it is fitted.

Then, according to the principle announced in the foregoing proposition and demonstrated above, we shall obtain

$$a : A :: p : P.$$

But because, by the principles of mensuration, the areas of different circles are to one another as the squares of their diameters; if therefore, we substitute d^2 and D^2 respectively for a and A in the above analogy, we shall have

$$d^2 : D^2 :: p : P,$$

and from this, by making the product of the mean terms equal to the product of the extremes, we get

$$pD^2 = Pd^2. \qquad (88).$$

122. This is the principle upon which depends the construction and use of that very powerful instrument, the *Hydrostatic Press,* first brought into notice about the year 1796, by Joseph Bramah, Esq., of Pimlico, London; who announced it to the world as the discovery of a new mechanical power.

In this however, he was mistaken, for although the principle upon which it depends may be said to constitute a seventh mechanical power, yet the principle announced in Proposition II. was not new to philosophers at the time when Mr. Bramah applied it to the construction of his presses, it having long been familiarly known under the designation of the *Hydrostatic Paradox;* and besides, the celebrated Pascal obscurely hinted at its application to mechanical purposes, but did not pursue the idea far enough to produce any thing useful, or to entitle him to the full merit of the discovery.

The improvement introduced by Mr. Bramah, consisted in the application of the common forcing pump to the injection of water, or some other incompressible and non-elastic fluid, into a strong metallic cylinder, truly bored and furnished with a moveable piston, made perfectly water-tight by means of leather collars or packing, neatly fitted into the cylinder.

123. The proportion which subsists between the diameter of this piston, and that of the plunger in the forcing pump, constitutes the principal element by which the power of the instrument is calculated; for, by reason of the equal distribution of pressure in the fluid, it is evident, that whatever force is applied, that force must operate alike on the piston in the cylinder, and on the plunger in the forcing pump, and consequently,

> *In proportion as the area of the transverse section of the one, exceeds the area of a similar section of the other, so must the pressure sustained by the one, exceed that sustained by the other.*

Therefore, if the piston F in the preceding diagram, be assimilated to the plunger in the barrel of a forcing pump, and the piston E to that in the cylinder of the hydrostatic press; then, the equation marked (88), notwithstanding the very simple and concise form in which it appears, involves every particular respecting the power and effects of the engine, of which a detailed description with illustrative drawings will be given a little further on.

This being premised, we shall now proceed to exhibit the use and application of the formula, by the resolution of the following practical examples.

124. EXAMPLE 1. If the diameter of the cylinder is 5 inches, and that of the forcing pump one inch; what is the pressure on the piston in the cylinder, supposing the force applied on the plunger or smaller piston, to be equivalent to 750 lbs.?

Here we have given $D = 5$ inches; $d = 1$ inch, and $p = 750$ lbs.; therefore, by substitution, equation (88) becomes

$$5^2 \times 750 = P \times 1^2; \text{ that is, } P = 18750 \text{ lbs.}$$

Or the equation for the value of P, may be expressed in general terms, as follows.

$$P = \frac{pD^2}{d^2}. \qquad (89).$$

And from the equation in its present form, we deduce the following practical rule.

> Rule. *Multiply the square of the diameter of the cylinder by the magnitude of the power applied, and divide the product by the square of the diameter of the forcing pump, and the quotient will express the intensity of the pressure on the piston of the cylinder.*

125. Example 2. If the diameter of the cylinder is 5 inches, and that of the forcing pump one inch; what is the magnitude of the power applied, supposing the entire pressure on the piston of the cylinder to be 18750 lbs.?

Here we have given $D = 5$ inches; $d = 1$ inch, and $P = 18750$ lbs.; therefore, by substitution, equation (88) becomes

$$5^2 \times p = 18750 \times 1^2; \text{ or } p = 750 \text{ lbs.}$$

If both sides of the fundamental equation (88) be divided by D^2, the general expression for the value of p, is

$$p = \frac{P d^2}{D^2}. \qquad (90).$$

And the practical rule which this equation supplies, may be expressed in words at length in the following manner.

> Rule. *Multiply the given pressure on the piston of the cylinder, by the square of the diameter of the forcing pump, and divide the product by the square of the diameter of the cylinder for the power required.*

126. Example 3. The diameter of the forcing pump is one inch, and the power with which the plunger descends is equivalent to 750 lbs.; what must be the diameter of the cylinder, to admit a pressure of 18750 lbs. on the piston?

Here we have given $d = 1$ inch; $p = 750$ lbs., and $P = 18750$ lbs.; consequently, by substitution, the equation marked (88) becomes

$$750 D^2 = 18750 \times 1^2;$$

hence, by division, we obtain

$$D^2 = \frac{18750}{750} = 25;$$

consequently, by evolution, we have

$$D = \sqrt{25} = 5 \text{ inches.}$$

If both sides of the equation (88) be divided by p, and the square root of the quotient extracted, the general expression for the diameter of the piston, is

$$\text{D}=\sqrt{\frac{\text{P}d^2}{p}}. \qquad (91).$$

And the practical rule for the determination of D, may be expressed in words as follows.

> RULE. *Multiply the pressure on the piston of the cylinder, by the square of the diameter of the forcing pump, and divide the product by the force with which the plunger descends; then, the square root of the quotient will be the diameter of the cylinder sought.*

127. EXAMPLE 4. The diameter of the cylinder is 5 inches, and the force with which the plunger descends, is equivalent to 750 lbs.; what must be the diameter of the forcing pump, in order to transmit a pressure of 18750 lbs. to the piston of the cylinder?

Here we have given $\text{D}=5$ inches; $p=750$ lbs., and $\text{P}=18750$ lbs.; consequently, by substitution, equation (88) becomes

$$18750\,d^2=750\times 5^2,$$

and by division, we shall have

$$d^2=\frac{750\times 25}{18750}=1;$$

therefore, by extracting the square root, we get

$$d=\sqrt{1}=1 \text{ inch.}$$

If both sides of the original equation marked (88), be divided by P, and the square root extracted, the entire pressure on the piston, the general expression for the value of d becomes

$$d=\sqrt{\frac{p\text{D}^2}{\text{P}}}. \qquad (92).$$

And the practical rule which this equation supplies, may be expressed in words in the following manner.

> RULE. *Multiply the force with which the plunger descends, by the square of the diameter of the cylinder, and divide the product by the entire pressure on the piston; then, extract the square root of the quotient for the diameter of the forcing pump.*

128. The foregoing is the theory of the Hydrostatic Press, as restricted to the consideration of the diameters of the cylinder and forcing pump, and the respective pressures on the piston and plunger; but since the instrument is generally furnished with an *indicator* or

safety valve for measuring the intensity of pressure, the theory would be incomplete without considering it in connection with the diameters of the pump and cylinder. For which purpose

Put $\delta =$ the diameter of the safety valve, expressed in inches or parts,

and $w =$ the weight thereon, or the force that prevents its rising.

Then, according to the principle announced in Proposition II., we obtain the following analogies, viz.

$$D^2 : \delta^2 :: P : w,$$
$$d^2 : \delta^2 :: p : w;$$

and from these analogies, by making the products of the extreme terms equal to the products of the means, we get

$$D^2 w = \delta^2 P, \qquad (93).$$
$$\text{and } d^2 w = \delta^2 p. \qquad (94).$$

Now, in order to pursue the expansion of these equations, we shall suppose the value of δ to be one fourth of an inch, while the numerical values of the other letters remain the same as supposed for the several examples under equation (88); then, to determine the corresponding value of w, or the power which prevents the safety valve from rising, when all the parts of the instrument, or the several powers and pressures are in a state of equilibrium, we have the following examples to resolve according to the proposed conditions.

129. EXAMPLE 5. The diameter of the cylinder is 5 inches, that of the indicator or safety valve $\frac{1}{4}$ of an inch, and the entire pressure upon the piston of the cylinder 18750 lbs.; what is the corresponding force preventing the ascent of the safety valve, on the supposition of a perfect equilibrium?

Here we have given $D = 5$ inches; $\delta = \frac{1}{4}$ of an inch, and $P = 18750$ lbs.; consequently, by substitution, the equation (93) becomes

$$5^2 w = .25^2 \times 18750;$$

from which, by division, we get

$$w = \frac{.0625 \times 18750}{25} = 46.875 \text{ lbs.}$$

But the general expression for the value of w, as derived from the equation (93), becomes

$$w = \frac{\delta^2 P}{D^2}. \qquad (95).$$

From which we derive the following rule.

> Rule. *Multiply the entire pressure on the piston of the cylinder, by the square of the diameter of the indicator or safety valve, and divide the product by the square of the diameter of the cylinder for the weight required.*

130. Example 6. The diameter of the safety valve is $\frac{1}{4}$ of an inch, that of the cylinder 5 inches, and the weight on the safety valve 46.875 lbs.; what is the corresponding pressure on the piston of the cylinder?

Here we have given $\delta = \frac{1}{4}$ of an inch; $D = 5$ inches, and $w = 46.875$ lbs; therefore, by substitution, equation (93) becomes

$$.25^2 P = 5^2 \times 46.875,$$

and by division, we obtain

$$P = \frac{1171.875}{.0625} = 18750 \text{ lbs.}$$

The general expression for the value of P, as derived from the equation marked (93), becomes

$$P = \frac{D^2 w}{\delta^2}. \qquad (96).$$

And the practical rule supplied by this equation, may be expressed in words as follows.

> Rule. *Multiply the weight on the safety valve, by the square of the diameter of the cylinder, and divide the product by the square of the diameter of the safety valve, and the quotient will give the entire pressure on the piston of the cylinder.*

131. Example 7. The diameter of the cylinder is 5 inches, the entire pressure of the piston is 18750 lbs., and the weight on the safety valve is 46.875 lbs.; what is its diameter?

Here we have given $D = 5$ inches; $P = 18750$ lbs., and $w = 46.875$ lbs.; therefore, by substitution, equation (93) becomes

$$18750\,\delta^2 = 5^2 \times 46.875,$$

and from this, by division, we get

$$\delta^2 = \frac{5^2 \times 46.875}{18750} = .0625.$$

and by extracting the square root, we obtain

$$\delta = \sqrt{.0625} = .25, \text{ or } \tfrac{1}{4} \text{ of an inch.}$$

The general expression for the value of δ, as derived from the equation (93), is as follows, viz.

$$\delta=\sqrt{\frac{D^2 w}{P}}. \qquad (97).$$

And the practical rule which this equation affords, may be expressed in words in the following manner.

> Rule. *Multiply the load on the safety valve by the square of the diameter of the cylinder; divide the product by the entire pressure on the piston, and the square root of the quotient will give the diameter of the safety valve required.*

132. Example 8. The diameter of the safety valve is $\frac{1}{4}$ of an inch, the load upon it 46.875 lbs., and the entire pressure on the piston of the cylinder is 18750 lbs.; what is its diameter?

Here we have given $\delta=\frac{1}{4}$ of an inch, $w=46.875$ lbs., and $P=18750$ lbs.; consequently, by substitution, we have

$$46.875\, D^2=.25^2\times 18750,$$

from which, by division, we shall obtain

$$D^2=\frac{.25^2\times 18750}{46.875}=25,$$

and finally, by extracting the square root, we get

$$D=\sqrt{25}=5 \text{ inches}.$$

If both sides of the equation marked (93), be divided by w the weight on the safety valve, we get

$$D^2=\frac{\delta^2 P}{w},$$

and by extracting the square root, the general expression for the value of D the diameter of the cylinder, becomes

$$D=\sqrt{\frac{\delta^2 P}{w}}. \qquad (98).$$

And from this equation we derive the following rule.

> Rule. *Multiply the entire pressure on the piston of the cylinder by the square of the diameter of the safety valve, divide the product by the weight upon the safety valve, and extract the square root of the quotient for the diameter of the cylinder sought.*

133. Example 9. The diameter of the forcing pump is one inch, that of the safety valve is one fourth of an inch, and the power or force with which the plunger descends, is equivalent to 750 lbs.; what is the corresponding weight on the safety valve?

Here we have given $d = 1$ inch; $\delta = \frac{1}{4}$ of an inch, and $p = 750$ lbs.; consequently, by substitution, the equation (94) becomes

$1^2 \times w = .25^2 \times 750$; that is, $w = 46.875$ lbs., the very same value as we derived from the fifth example.

If both sides of the equation marked (94) be divided by d^2, the general expression for the value of w becomes

$$w = \frac{\delta^2 p}{d^2}. \tag{99}$$

And the practical rule supplied by this equation, may be expressed in words at length in the following manner.

> RULE. *Multiply the force with which the plunger descends by the square of the diameter of the safety valve, and divide the product by the square of the diameter of the plunger; then the quotient will express the load upon the safety valve.*

134. EXAMPLE 10. The diameter of the safety valve is $\frac{1}{4}$ of an inch, that of the forcing pump is one inch, and the load upon the safety valve is 46.875 lbs.; what is the power applied, or the force with which the plunger in the forcing pump descends?

Here we have given $\delta = \frac{1}{4}$ of an inch, $d = 1$ inch, and $w = 46.875$ lbs.; consequently, by substitution, equation (94) becomes

$$.25^2 p = 46.875 \times 1^2,$$

and from this, by division, we obtain

$$p = \frac{46.875}{.0625} = 750 \text{ lbs.}$$

The general expression for the value of p, as obtained from the equation marked (94), becomes

$$p = \frac{d^2 w}{\delta^2}. \tag{100}$$

from which we derive the following rule.

> RULE. *Multiply the load on the safety valve by the square of the diameter of the forcing pump; then, divide the product by the square of the diameter of the safety valve, and the quotient will give the force with which the piston descends.*

135. EXAMPLE 11. The diameter of the plunger or the piston of the forcing pump is one inch, the force with which it descends is equivalent to 750 lbs., and the load on the safety valve is 46.875 lbs.; what is its diameter?

Here we have given $d = 1$ inch, $p = 750$ lbs., and $w = 46.875$ lbs.; consequently, by substitution, we have

$$750\delta^2 = 1^2 \times 46.875,$$

and from this, by division, we obtain

$$\delta^2 = \frac{46.875}{750} = .0625,$$

and finally, by evolution, we have

$$\delta = \sqrt{.0625} = .25 \text{ of an inch.}$$

Let both sides of the equation marked (94) be divided by p, the power or force with which the piston of the forcing pump descends, and we shall have

$$\delta^2 = \frac{d^2 w}{p},$$

and by extracting the square root, we get

$$\delta = \sqrt{\frac{d^2 w}{p}}. \tag{101}$$

Hence, the following practical rule.

RULE. *Multiply the weight or load upon the safety valve, by the square of the diameter of the forcing pump, and divide the product by the force with which the plunger or piston of the forcing pump descends; then, the square root of the quotient will be the diameter of the safety valve.*

136. EXAMPLE 12. The diameter of the safety valve is one fourth of an inch, the weight upon it is 46.875 lbs., and the power applied, or the force with which the plunger descends, is 750 lbs; what is the diameter of the forcing pump?

Here we have given $\delta = \frac{1}{4}$ of an inch, $w = 46.875$ lbs., and $p = 750$ lbs.; consequently, by substitution, the equation marked (94) becomes

$$46.875 d^2 = .25^2 \times 750;$$

therefore, by division, we obtain

$$d^2 = \frac{.25^2 \times 750}{46.875} = 1,$$

and finally, by extracting the square root, we get

$$d = 1 \text{ inch.}$$

The general expression for the value of the diameter of the forcing pump, as derived from the equation (94), is

$$d = \sqrt{\frac{\delta^2 p}{w}}. \tag{102}$$

And from this, we obtain the following practical rule.

RULE. *Multiply the force with which the piston of the forcing pump descends, by the square of the diameter of the safety valve; divide the product by the load on the safety valve, and extract the square root of the quotient for the diameter of the forcing pump.*

The foregoing twelve examples exhibit all the varieties of cases that can arise, from the combination of the six data which we have employed in our theory, viz. the diameters of the cylinder, the forcing pump and the safety valve; together with the entire pressure on the piston of the cylinder, the power applied to the plunger of the forcing pump, and the weight upon the safety valve.

We have determined each of the quantities, composing the several fundamental equations, in terms of the others, and have drawn up rules from the general expressions, merely for the assistance of those who are not accustomed to algebraic reductions; those who are, will prefer finding each quantity directly from the general equation expressing its value.

137. It is manifest from the principles of mensuration, that the area of a transverse section of the cylinder, or the base of the piston, is expressed by $.7854\,\mathrm{D}^2$; and we have shown, equations (89) and (96), that the entire pressure upon the base of the piston in the case of equilibrium, is

$$\mathrm{P}=\frac{p\,\mathrm{D}^2}{d^2},\text{ and }\mathrm{P}=\frac{\mathrm{D}^2 w}{\delta^2}; \qquad (103).$$

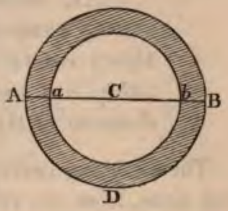
consequently, if n denotes the pressure in pounds avoirdupois on one square inch of the piston, then we have

$$n=\frac{\mathrm{P}}{.7854\,\mathrm{D}^2};\ n=\frac{p}{.7854\,d^2},\text{ and }n=\frac{w}{.7854\,\delta^2}. \qquad (104).$$

Now, from principles investigated by Professor Barlow, of the Royal Military Academy at Woolwich, it appears, that if c denote the cohesive force of the material employed in the construction of the cylinder, t its thickness, and r the interior radius; then, in order that the strain produced by the pressure, shall not exceed the elastic power of the material; it is necessary that

$$n=\frac{ct}{t+r}.$$

In order to demonstrate this, let ABD be a transverse section of the cylinder, perpendicular to the axis passing through C; then, supposing a certain uniform pressure to be exerted all round the interior boundary; it will readily appear, from the theory of resistance, that each successive circular lamina, estimated from the interior towards the exterior circumference, offers a less and less resistance to the straining force.



But it is obvious from the very nature of the subject, that by reason of the internal pressure or strain, the metal must undergo a certain degree of extension, and since the resistance of the outer boundary is less than that of the inner one, it follows, that the extension must also be less; this is manifest, for the resistance which any body offers to the force by which it is strained, is proportional to the extension which it undergoes, divided by its length; now, since the resistances of the several laminæ, decrease as they recede from the interior boundary towards the exterior, while at the same time, the corresponding circumferences increase; it is manifest, that the extension for the several laminæ decreases to the last or exterior boundary, where it is the least of all :—It is therefore the law of the decreasing resistance, that the present enquiry is instituted to determine.

Put $d = ab$, the interior diameter of the cylinder before the pressure is applied,
$e =$ the increase of d occasioned by the pressure,
$d' =$ A B, the exterior diameter in its original state,
$e' =$ the increase induced by pressure.

Then $(d + e)$ and $d' + e'$), are respectively the interior and exterior diameters of the cylinder as affected by extension.

By the principles of mensuration, the area of the annulus, or circular ring contained between the interior and exterior boundaries :—

> *Is equal to the difference of the squares of the diameters, drawn into the constant fraction* 0.7854; *or it is proportional to the sum of the diameters, drawn into their difference.*

But according to the nature of the present enquiry, the area of the ring is the same, both before and after the extension takes place; consequently, we have

$$(d' + e')^2 - (d + e)^2 = d^2 - d^2;$$

therefore, by expanding the terms on the left hand side, we get

$$d'^2 + 2d'e' + e'^2 - d^2 - 2de - e^2 = d^2 - d^2;$$

or by transposing and expunging the common terms, it is

$$2d'e' + e'^2 = 2de + e^2;$$

and this equation being converted into an analogy, gives

$$2d' + e' : 2d + e :: e : e'.$$

Now, the quantity of extension that the material will allow before rupture being very small, especially as compared with the quantities $2d'$ and $2d$; it therefore follows, that the quantities e' and e, in the first and second terms, may be conceived to vanish, and the above analogy becomes

$$d' : d :: e : e'.$$

From this it appears, that the extensions of the respective circumferences, are inversely as the corresponding diameters; but we have stated above, that the resistance is as the extension divided by the length; therefore, we have

$$\frac{d}{d'} : \frac{d'}{d},$$

or which amounts to the same thing,

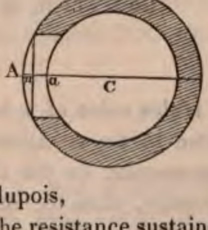
$$d^2 : d'^2;$$

hence this law, that the magnitude of the resistance offered by each successive circular lamina:—

> *Is inversely as the square of its diameter, or, which is the same thing, inversely as the square of its distance from the common centre to which they are referred.*

From the general law thus established, the actual resistance due to any point in the annulus, or to any thickness of metal, can very easily be ascertained.

Put $r = \mathrm{C}a$, the interior radius of the cylinder, of which the annexed diagram is a section,



$t = a\mathrm{A}$, the entire thickness of the metal,

$x = an$, any variable thickness estimated from a, the interior surface,

$n =$ the pressure on a square inch of the inner surface in pounds avoirdupois,

$f =$ the measure of the straining force, or the resistance sustained by the first or interior lamina, and

$c =$ the cohesive force of the material.

Then, agreeably to the law of the resistances which we have established above, we have

$$(r+x)^2 : r^2 :: f : \frac{fr^2}{(r+x)^2};$$

this result expresses the strain at the point x, or the resistance of the material whose thickness is an; and the fluxion of this quantity as referred to the variable thickness x, is

$$\text{flux.} = \frac{fr^2\dot{x}}{(r+x)^2};$$

consequently, the fluent, or the sum of all the strains, is

$$\int \frac{fr^2\dot{x}}{(r+x)^2} + \mathrm{C}, \text{ and this when } x = t \text{ becomes}$$

$$f\left(r - \frac{r^2}{r+t}\right) = \frac{frt}{r+t}.$$

Therefore, if the strain or resistance f, were to act uniformly on the thickness expressed by $\frac{frt}{r+t}$, it would produce the same effect, as if all the variable strains were to act on the whole thickness t.

The above law being admitted, let us suppose that the interior radius of the cylinder, and the pressure per square inch on the surface are given, and let it be required to determine the thickness such, that the strain and resistance may be in equilibrio.

Here it is manifest, that the greatest strain the thickness $\frac{rt}{r+t}$ can resist, is $\frac{crt}{r+t}$, and the strain to which it is actually exposed, is nr; consequently, when these are equal, we have

$$nr = \frac{crt}{r+t};$$

from which, by expunging the common factor r, we get

$$n = \frac{ct}{r+t}. \qquad (105).$$

If this value of n be compared with its respective values, as indicated in the equations (104) preceding, we shall have the following expressions, for the thickness of metal in the cylinder to resist any pressure, while the elastic power of the material remains perfect, viz.

$$t = \frac{\mathrm{P}r}{.7854c\mathrm{D}^2 - \mathrm{P}};\quad t = \frac{pr}{.7854cd^2 - p}, \text{ and } t = \frac{wr}{.7854cd^2 - w}.$$

Therefore, if for c in each of the preceding expressions, we substitute its value as determined by experiment, and which for cast iron, according to Dr. Robison, is 16648 pounds avoirdupois upon a square inch; then we shall have

$$t = \frac{\mathrm{P}r}{13076\,\mathrm{D}^2 - \mathrm{P}}.^{*} \qquad (106).$$

$$t = \frac{pr}{13076 d^2 - p}. \qquad (107).$$

$$t = \frac{wr}{13076 \delta^2 - w}. \qquad (108).$$

The following example will illustrate the use of these equations, the value of t the thickness of the metal coming out the same by each.

138. EXAMPLE 13. What must be the thickness of metal in the cylinder of a Hydrostatic Press, to resist a pressure of 30000 lbs.; the diameter of the cylinder being 5 inches, that of the forcing pump one inch, and of the safety valve one fourth of an inch; being the same dimensions which we have employed in the preceding examples?

Here we have given $\mathrm{P} = 30000$ lbs.; $\mathrm{D} = 5$ inches; and consequently, $r = 2\frac{1}{2}$ inches; therefore, by substitution, equation (106) gives

$$t = \frac{30000 \times 2\frac{1}{2}}{13076 \times 5^2 - 30000} = .253 \text{ inches, being some-}$$

thing more than one fourth of an inch.

In order that the entire pressure on the piston of the cylinder, may be equal to 30000 lbs. according to the conditions of the question; the force with which the plunger of the forcing pump descends, must be equal to 1200 lbs.; therefore, by equation (107), we have

$$t = \frac{1200 \times 2\frac{1}{2}}{13076 \times 1^2 - 1200} = .253 \text{ inches, the same as}$$

before.

Again, in order that the entire pressure may be equal to 30000 lbs. the weight upon the safety valve must be 75 lbs.; hence, from equation (108), we obtain

$$t = \frac{75 \times 2\frac{1}{2}}{13076 \times .25^2 - 75} = .253 \text{ inches, the same as in}$$

the two cases foregoing.

139. It may not be improper here to remark, that although the requisite thickness of metal is alike assignable from either of the above equations, when the respective pressure and diameters are

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[*] Where the constant number $13076 = 16648 \times .7854$.

known; yet it is the first of the class only, or that marked (106), which becomes available in practice, and for this reason, that the power of the press, or the aggregate pressure which it is capable of exciting, is known *à priori*, or immediately assignable from the conditions of construction, while the load upon the safety valve, and the force with which the plunger descends, have each to be determined by calculations founded on circumstances connected with the aggregate or ultimate pressure.

140. Referring to equation (105), which has been purposely investigated, for expressing the intensity of pressure on a square inch of surface, and multiplying both sides by $r + t$ the denominator of the fraction, we shall have

$$nr + nt = ct,$$

from which, by transposing and collecting the terms, we get

$$(c - n)\,t = nr;$$

then by division, the value of t, or the thickness of metal in the cylinder to withstand the pressure, becomes

$$t = \frac{nr}{c - n}. \qquad (109).$$

From which it appears, that if a constant value adapted to practical purposes, can be assigned to n, the rule for calculating the thickness of metal in the cylinder will become exceedingly simple.

Now, it has been remarked by several eminent practical engineers, as well as by the most approved and intelligent manufacturers, that the extreme pressure on a square inch of the piston,* should never exceed half the cohesive power of the material; but according to Dr. Robison, the cohesive power of cast iron of a medium quality is equal to 16648 lbs.; hence we have

$$n = \frac{16648}{2} = 8324 \text{ lbs.};$$

therefore, if 8324 lbs. be adopted as the limit of pressure upon a square inch of surface, the foregoing value of t becomes

$$t = \frac{8324\,r}{16648 - 8324} = r;$$

consequently, in order that the strain produced by the pressure may not exceed the elastic power of the material;—

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* There is no occasion to limit the pressure to the piston only, since every square inch of surface in contact with the fluid sustains the same pressure. This limitation has frequently caused a misapprehension respecting the mode of ascertaining the pressure on an inch of surface.

The thickness of metal ought never to be less than the interior radius of the cylinder.

By the first equation of class (104), it has been shown, that the pressure on a square inch of the piston in lbs. avoirdupois, is

$$n = \frac{P}{.7854\,D^2},$$

or by substituting the foregoing value of n, it is

$$8324 = \frac{P}{.7854\,D^2};$$

from which, by multiplication, we obtain

$$8324 \times .7854\,D^2 = P;$$

but in order to express the pressure in tons, it is

$$P = \frac{6537.6696\,D^2}{2240} = 2.9186\,D^2. \qquad (110).$$

141. Therefore, when the diameter of the cylinder is given, the entire pressure in tons is determined by the following very simple rule.

Rule. *Multiply the square of the diameter in inches, by the constant number* 2.9186, *and the product will be the pressure in tons.*

And again, when the pressure in tons is given, the diameter of the cylinder may be determined by reversing the process, or by the following rule.

Rule. *Divide the given pressure in tons by the constant number* 2.9186, *and extract the square root of the quotient, for the diameter of the cylinder in inches.*

142. The preceding theory, as we have developed it, unfolds every particular connected with the Hydrostatic Press, and by paying proper attention to the equations, rules, and examples, as we have delivered them, every difficulty attending the construction of the instrument will be removed; to practical persons, however, that part of the theory exhibited in the equations marked (109) and (110) will be found the most valuable, as they do the more immediately contain the particulars which direct their operations. The following examples will prove the truth of these remarks.

Example 14. The diameter of the cylinder in a Hydrostatic Press, is 10 inches; what is its power, or what pressure does it transmit?

Here by the first rule above, we have

$$P = 10^2 \times 2.9186 = 291.86 \text{ tons}.$$

EXAMPLE 15. What is the diameter, and what the thickness of metal, in a press of 300 tons power?

By the second rule above, we have

$$D^2 = 300 \div 2.9186 = 102.81 \text{ nearly};$$

therefore, by extracting the square root, we obtain

$$D = \sqrt{102.81} = 10.13 \text{ inches};$$

consequently, according to the remark under the equation (109), the thickness of metal is

$$t = 10.13 \div 2 = 5.065 \text{ inches}.$$

143. The rules by which the preceding examples have been resolved, are very nearly, but not precisely the same as those employed by Messrs. Bramah in the construction of their excellent presses; the only difference, however, consists in their assuming a higher number as the limit of pressure, the standard which they employ being 8556 lbs. upon a square inch of the piston, thereby indicating, that they reckon on a higher cohesive power in the material, than that which we have adopted as the basis of our theory.

Now, 8556 lbs. on a square inch, is equivalent to 6619.8824 lbs. upon a circular inch; whereas the constant which we have chosen is only 6537.6696 lbs., being a difference of 82.2128 lbs. upon the circular inch, a difference that need not be regarded in practice, as the error will always fall on the side of safety, giving a smaller power to the press than what it really possesses.

144. It sometimes, indeed it very frequently happens, that presses are constructed, without any attention being paid, to the relation which subsists between the strength of the parts, and the strain which they have to resist; in all such cases, therefore, it may be interesting to possess a rule, by which the merits or demerits of a press so constructed can be ascertained, for in this way a failure in the instrument may be prevented, and a remedy applied to any defect that may exist.

Now, according to the first equation of class (104), the pressure upon a square inch is

$$n = \frac{P}{.7854 D^2},$$

and according to equation (105), it is

$$n = \frac{ct}{r+t};$$

therefore, by comparison, we have

$$\frac{\mathrm{P}}{.7854\,\mathrm{D}^2}=\frac{ct}{r+t};$$

consequently, by multiplying and substituting the cohesive power of cast iron, we have

$$(t+r)\,\mathrm{P}=13076\,\mathrm{D}^2t. \qquad (111).$$

Let $4r^2$, be substituted in this equation, instead of D^2 its equivalent, and we shall obtain

$$(t+r)\,\mathrm{P}=52304\,r^2t;$$

consequently, the pressure in tons, is

$$\mathrm{P}=\frac{52304\,r^2t}{2240(t+r)}=\frac{23.35\,r^2t}{(t+r)}. \qquad (112).$$

From which it appears, that by knowing the interior radius of the cylinder and the thickness of the metal, the power of the press can easily be ascertained; the following is the rule for that purpose.

> RULE. *Multiply* 23.35 *times the thickness of metal by the square of the radius of the cylinder, and divide the product by the radius plus the thickness of metal, and the quotient will give the power of the press in tons.*

145. EXAMPLE 16. A Hydrostatic Press is so constructed, as to have the interior radius of its cylinder equal to 3 inches, and the thickness of metal 4 inches; now this press is designed for packing flax, and is estimated to stand a pressure of 180 tons; query if its power is not overrated?

According to the above rule, it is

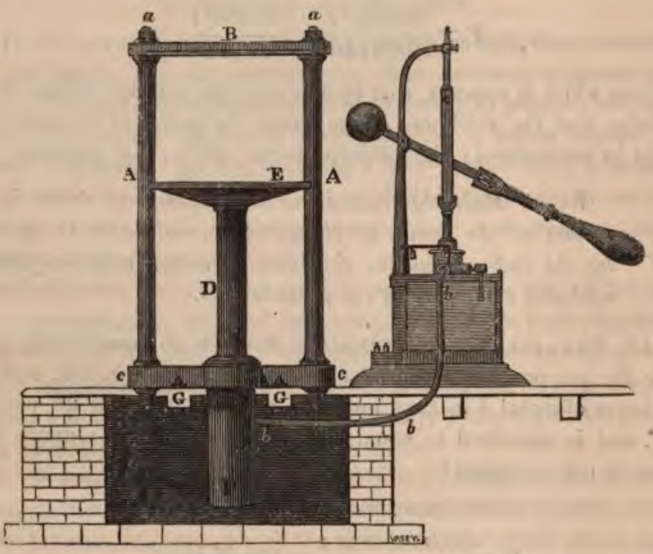
$$\mathrm{P}=\frac{23.35\times3^2\times4}{4+3}=120.08 \text{ tons};$$

consequently, the power of the press is overrated by about 60 tons, being one third less than the estimated pressure according to the question.

The thickness of metal necessary to resist a pressure of 180 tons or 403200 lbs. is equal to 17.9 inches nearly, and the proposed thickness is only 4 inches, being less than one fourth of the thickness which is really necessary to resist the strain; hence we infer that the press in its present state, is entirely unfitted for its intended purpose, and altogether inconsistent with safety and precision of operation. Here follows the description of a press when completely furnished in all its parts and fit for immediate action.

146. The *Hydrostatic Press*, in its present high and refined state of improvement, is a machine that is capable of generating and transmitting a greater degree of force, for the purpose of overcoming immense resistances, and raising enormous loads to a small height, than any other instrument or engine with which we are acquainted; it is therefore of the highest importance that the principles of its construction and the mode of operation should be rightly understood, and in order to render the subject as clear and intelligible as possible, we think proper to lay before our readers the following detailed description.

Fig. 1.



The wood cut before us, *fig.* 1, exhibits an *elevation* of the press in its complete state, accompanied by the forcing pump and all its appurtenances as fitted up for immediate action: F is a strong metallic cylinder of cast iron, or some other material of sufficient density to prevent the fluid from issuing through its pores, and of sufficient strength to preclude the possibility of rupture, by reason of the immense pressure which it is destined to withstand.

The cylinder F is bored and polished with the most scrupulous precision, and fitted with the moveable piston D, which is rendered perfectly water-tight, by means of leather collars constructed for the purpose, and fixed in the cylinder by a simple but ingenious contrivance to be described hereafter.

Into the side or base of the cylinder F, the end of a small tube *b b b* is inserted, and by this tube the water is conveyed or forced into the cylinder; the other end of the tube is attached to the forcing pump, as represented in the diagram; but this will be more particularly explained in another place.

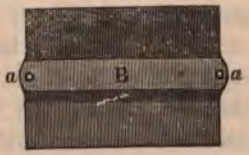
A A are two very strong upright bars, generally made of wrought iron, and of any form whatever, corresponding to the notches in the sides of the flat table E, which is fixed upon the end of the piston D, and by workmen, is usually denominated the '*Follower*' or '*Pressing Table*.'

B is the top of the frame into which the upright bars A A are fixed, and *c c* is the bottom thereof, both of which are made of cast, in preference to wrought iron, being both cheaper and more easily moulded into the intended form.

The bottom of the frame *c c*, is furnished with four projections or lobes, with circular perforations, for the purpose of fastening it by iron bolts to the massive blocks of wood, whose transverse sections are indicated by the lighter shades at G G. The top B has two similar perforations, through which are passed the upper extremities of the vertical bars A A, and there made fast, by screwing down the *cup-nuts* represented at *a* and *a*.

Fig. 2 represents the plan of the top, or as it is more frequently termed, the head of the frame; the lower side or surface of which is made perfectly smooth, in order to correspond with, and apply to the upper surface of the pressing table E in *fig.* 1; this correspondence of surfaces becomes necessary on certain occasions, such as the copying of prints, taking fac-similes of letters and the like; in all such cases, it is manifest, that smooth and coincident surfaces are indispensable for the purpose of obtaining true impressions.

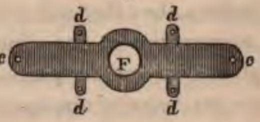
Fig. 2.



The figure before us represents the upper side of the block, where it is evident, that the middle part B, (through whose rounded extremities *a* and *a*, the circular perforations are made for receiving the upright bars or rods A A, *fig.* 1), is considerably thicker than the parts on each side of it; this augmentation of thickness, is necessary to resist the immense strain that comes upon it in that part; for although the pressure may be equally distributed throughout the entire surface, yet it is obvious, that the mechanical resistance to fracture, must principally arise from that part, which is subjected to the re-action of the upright bars.

Fig. 3 represents the plan of the base or bottom of the frame; it is generally made of uniform thickness, and of sufficient strength to withstand the pressure, for be it understood, that all the parts of the machine are subjected to the same quantity of strain, although it is exerted in different ways.*

Fig. 3.

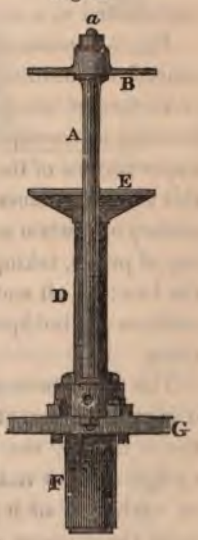


The circular perforations *cc* correspond to *aa* in the top of the frame, and receive the upright bars in the same manner; the perforations *dddd*, receive the screw bolts which fix the frame to the beams of timber represented at G G, *fig.* 1; the large perforation F receives the cylinder, the upper extremity of which is furnished with a flanch, for the purpose of fitting the circular swell around the perforation, and preventing it from moving backwards during the operation of the instrument.

When the several parts which we have now described are fitted together, they will present us with that portion of the drawing in *fig.* 1 denominated "*Elevation of the Press.*"

A side view of the engine as thus completed, is represented in *fig.* 4, where, as is usual in all such descriptions, the same letters of the alphabet refer to the same parts of the structure.

Fig. 4.



F is the cylinder into which the fluid is injected; D the piston, on whose summit is the pressing table E; A one of the upright rods or bars of malleable iron; B the head of the press, fixed to the upright bar A by means of the *cup-nut a*; *c* the bottom, in which the upright bar is similarly fixed; and G a beam of timber supporting the frame with all its appendages.

147. But the *Hydrostatic Press* as here described and constructed, must not be considered as fit for immediate action; for it is manifestly impossible to bore the interior of the cylinder so truly, and to turn the piston

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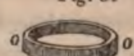
* The upright bars, cylinders, and connecting tubes, resist by tension, the pistons by compression, and the pressing table, together with the top and bottom of the frame, resist transversely.

with so much precision, as to prevent the escape of water between their surfaces, without increasing the friction to such a degree, that it would require a very great force to counterbalance it.

In order, therefore, to render the piston water-tight, and to prevent as much as possible the increase of friction, recourse must be had to other principles, which we now proceed to explain.

The piston D is surrounded by a collar of pump leather *oo*, represented in *fig.* 5, which collar being doubled up, so as in some measure to resemble a lesser cup placed within a greater, it is fitted into a cell made for its reception in the interior of the cylinder; and when there, the two parts are prevented from coming together, by means of the copper ring *pp*, represented in *fig.* 6, being inserted between the folds, and retained in its place, by a lodgement made for that purpose on the interior of the cylinder.

Fig. 5.



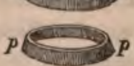
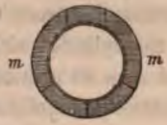


Fig. 6.

The leather collar is kept down by means of a brass or bell-metal ring *mm*, *fig.* 7, which ring is received into a recess formed round the interior of the cylinder, and the circular aperture is fitted to admit the piston D to pass through it, without materially increasing the effects of friction, which ought to be avoided as much as possible.

Fig. 7.



The leather is thus confined in a cell, with the edge of the inner fold applied to the piston D, while the edge of the outer fold is in contact with the cylinder all around its interior circumference; in this situation, the pressure of the water acting between the folds of the leather, forces the edges into close contact with both the cylinder and piston, and renders the whole water-tight; for if the leather be properly constructed and rightly fitted into its place, it is almost impossible that any of the fluid can escape; for the greater the pressure, the closer will the leather be applied to both the piston and the cylinder.

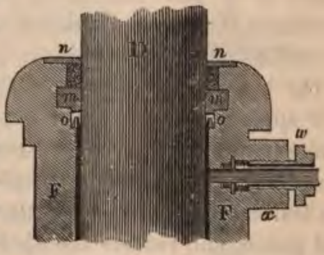
The metal ring *mm* is truly turned in a lathe, and the cavity in which it is placed is formed with the same geometrical accuracy; but in order to fix it in its cell, it is cut into five pieces by a very fine saw, as represented by the lines in the diagram, which are drawn across the surface of the ring. The four segments which radiate to the centre are put in first, then the segment formed by the parallel kerfs, (the copper ring *pp* and the leather collar *oo* being previously introduced), and lastly, the piston which carries the pressing table.

That part of the cylinder above the ring *mm*, where the inner surface is not in contact with the piston, is filled with tow, or some other soft material of a similar nature; the material thus inserted has a twofold use; in the first place, when saturated with sweet oil, it diminishes the friction that necessarily arises, when the piston is forced through the ring *mm*; and in the second place, it prevents the admission of any extraneous substance, which might increase the friction or injure the surface of the piston, and otherwise lessen the effects of the machine.

The packing here alluded to, is confined by a thin metallic annulus, neatly fitted and fixed on the top of the cylinder, the circular orifice being of sufficient diameter, to admit of a free and easy motion to the piston.

If a cylinder thus furnished with its several appendages be placed in the frame, and the whole firmly screwed together, and connected with the forcing pump, as represented in *fig*. 1, the press is completed and ready for immediate use; but in order to render the construction still more explicit and intelligible, and to show the method of connecting the press to the forcing pump, let *fig*. 8 represent a section of the cylinder with all its furniture, and a small portion of the tube immediately adjoining, by which the connexion is effected.

Fig. 8.



Then is FF the cylinder; D the piston; the unshaded parts *oo* the leather collar, in the folds of which is placed the copper ring *pp*, distinctly seen but not marked in the figure; *mm* is the metal ring by which the leather collar is retained in its place; *nn* the thin plate of copper or other metal fitted to the top of the cylinder, between which and the plate *mm* is seen the soft packing of tow, which we have described above, as performing the double capacity of oiling the piston and preventing its derangement.

The combination at *wx*, represents the method of connecting the injecting tube to the cylinder: it may be readily understood by inspecting the figure; but in order to remove all causes of obscurity, it may be explained in the following manner.

The end of the pipe or tube, which is generally made of copper, has a projecting piece or socket flanch soldered or screwed upon it, which fits into a perforation in the side or base of the cylinder, according to the fancy of the projector, but in the figure before us the perforation is in the side.

The tube thus furnished, is forcibly pressed into its seat by a hollow screw *w*, called an union screw, which fits into another screw of equal thread made in the cavity of the cylinder; the joint is made water-tight, by means of a collar of leather, interposed between the end of the tube and the bottom of the cavity.

A similar mode of connection is employed in fastening the tube to the forcing pump, the description of which, although it constitutes an important portion of the apparatus, does not properly belong to this place; the principles of its construction and mode of action, must therefore be supposed as known, until we come to treat of the construction and operation of pumps in general.

Admitting therefore, that the action of the forcing pump is understood, it only now remains to explain the nature of its operation in connection with the *Hydrostatic Press*, the construction of which we have so copiously exemplified.

148. In order to understand the operation of the press, we must conceive the piston D *fig.* 1, as being at its lowest possible position in the cylinder, and the body or substance to be pressed, placed upon the crown or pressing table E; then it is manifest, that if water be forced along the tube *b b b* by means of the forcing pump, it will enter the chamber of the cylinder F immediately beneath the piston D, and cause it to rise a distance proportioned to the quantity of fluid that has been injected, and with a force, determinable by the ratio between the square of the diameter of the cylinder and that of the forcing pump. The piston thus ascending, carries its crown, and consequently, the load along with it, and by repeating the operation, more water is injected, and the piston continues to ascend, till the body comes into contact with the head of the frame B, when the pressure begins; thus it is manifest, that by continuing the process, the pressure may be carried to any extent at pleasure; but we have already stated, in developing the theory, that there are limits, beyond which, with a given bore and a given thickness of metal, it would be unsafe to continue the strain.

When the press has performed its office, and it becomes necessary to relieve the action, the discharging valve, placed in the furniture of the forcing pump, must be opened, which will admit the water to escape out of the cylinder and return to the cistern, while the table and piston, by means of their own weight, return to their original position.

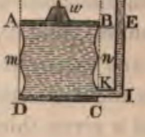
The method of calculating the power of the press, as well as every other particular respecting it, has been fully exemplified in the foregoing theory; it is hence unnecessary to dwell longer on the subject: we shall therefore conclude our description of the press, and proceed with that of the *Hydrostatic Bellows*, which depends upon the same principle, viz. the *quaqua versum* pressure of non-elastic fluids.

2. THEORY OF CONSTRUCTION AND SCIENTIFIC DESCRIPTION OF THE HYDROSTATIC BELLOWS.

149. In the preceding pages we have developed the theory and exemplified the application of the hydrostatic press; and furthermore, in order to render the subject as complete as possible, we have given a minute and comprehensive description of its several parts, and for the purpose of guiding the practical mechanic in its erection, the instrument is exhibited in its complete and finished state, accompanied by the forcing pump and all its requisite appendages.

The next subject, therefore, that claims our attention, is the *Hydrostatic Bellows*, an instrument of very frequent occurrence in philosophical experiments; it is chiefly employed in illustrating the upward pressure of non-elastic fluids and the hydrostatic paradox, and consequently, it depends upon the same principle as the hydrostatic press, admitting of a similar, but a more concise mode of discussion and illustration.

150. *The Hydrostatic Bellows* consists of a tube or pipe FEI, of very small diameter, and of any convenient length at pleasure, connected by means of the elbow at I, with a cylindrical vessel whose vertical section is CDGH, and whose sides are made of leather like a common bellows, represented by the waving lines A*m*D and B*n*K; the upper and the lower surfaces AB and DC, being formed of circular boards corresponding to the cylindrical form of the vessel.



When the bellows is empty, it is manifest that the boards AB and DC are very nearly in contact, and would be completely so, but for the leather sides forming into folds and preventing a coincidence: in this state, when water or any other incompressible and non-elastic fluid is poured into the tube, it flows into the bellows and separates the boards; a heavy weight as *w* is then placed upon the upper board, and by pouring more fluid into the tube, the moveable plane AB and its incumbent load w, will be raised and kept in equilibrio by the column of fluid in the tube; and when the equilibrium obtains, we infer, that:—

> *The weight of the supporting column of fluid in the tube, is to the weight upon the moveable plane, as the area of a section of the tube, is to the area of the plane.*

This is manifest, for the fluid at I, the lowest point of the vertical tube FEI, is pressed by a force varying as the altitude LI, and by the nature of fluidity, this pressure is communicated horizontally to all the particles in DC, and thence transmitted throughout the whole mass of fluid in the bellows; consequently, the pressure upwards on the board AB, is equal to the weight of a column of the fluid, the diameter of whose base is DC, and altitude LI or GD; but the actual weight of the fluid supported, is that of a column whose diameter is DC, and altitude EI or AD.

Hence, the weight which maintains the equilibrium, will be that of a cylinder of fluid, whose base is AB and altitude AG; consequently, the weight w, placed upon the moveable plane of the bellows, since it balances the column of fluid LE, is equivalent to the weight of a fluid cylinder, whose section along the axis is ABHG.

Put $D =$ AB or DC, the diameter of the cylindrical vessel or bellows,
$d =$ LM, the diameter of the vertical tube,
$w =$ the weight upon the moveable plane, and
$w' =$ the weight or pressure of the fluid in the column LE.

Then, because by the principles of mensuration, the areas of circles are to one another as the squares of their diameters; the foregoing inference gives

$$w' : w :: d^2 : D^2,$$

and this, by equating the products of the extreme and mean terms, becomes

$$D^2 w' = d^2 w. \qquad (113).$$

Let both sides of this equation be divided by the quantity D^2, which is found in combination with the weight or pressure of the fluid in the tube, and we shall obtain

$$w' = \frac{d^2 w}{D^2}. \qquad (114).$$

Here again, that singular property of non-elastic and incompressible fluids becomes manifest, viz.

That any quantity however small, may be made to balance any other quantity however great.

151. If the diameter of the tube, the diameter of the cylinder or bellows, and the weight upon the moveable board A B be given, the weight of the fluid in the tube, or its perpendicular altitude to maintain the equilibrium, can easily be determined by means of the equation (114), which affords the following practical rule.

RULE. *Multiply the square of the diameter of the tube by the load upon the moveable board, and divide the product by the square of the diameter of the bellows or cylinder; then, the quotient will give the weight of the fluid by which the equilibrium is maintained.*

EXAMPLE. The diameter of the bellows or cylindrical vessel is 18 inches, that of the tube or pipe, through which the fluid is conveyed into the vessel, is one fourth of an inch, and the weight upon the moveable board is 5760 lbs.; what weight of water must be poured into the vertical tube, so that the whole may remain at rest?

In this example there are given, $\mathrm{D} = 18$ inches; $d = \frac{1}{4}$ of an inch, and $w = 5760$ lbs.; therefore, by performing as directed in the rule, we shall have

$$w' = \frac{.25^2 \times 5760}{18^2} = \frac{360}{324} = 1\tfrac{1}{9} \text{ lbs.}$$

Here it appears, that a quantity of water weighing $1\frac{1}{9}$ lbs., disposed in a tube of $\frac{1}{4}$ of an inch in diameter, is capable of balancing another quantity of 5760 lbs., disposed in a cylinder of 18 inches diameter; it is therefore manifest, that the height of the one column must far exceed the height of the other, and the excess of altitude may be determined in the following manner.

152. It has been abundantly proved by experiment, that a cubic foot of distilled water, at the temperature of about 39° of Fahrenheit's Thermometer, weighs very nearly 1000 avoirdupois ounces, or $62\frac{1}{2}$lbs.; consequently, the number of cubic inches in the column whose weight is $1\frac{1}{9}$ lbs., is found by the following analogy, viz.

$$62\tfrac{1}{2} : 1728 :: 1\tfrac{1}{9} : 30\tfrac{18}{25} \text{ inches;}$$

hence, the solidity of a column which maintains the equilibrium is $30\frac{18}{25}$ inches, and according to the conditions of the question, the diameter of its base or section, is one fourth of an inch, and consequently, the area of the base or section, is

$$.25^2 \times .7854 = .0490875 \text{ square inches.}$$

Now, according to the principles of mensuration, the solidity of a cylinder is determined, by multiplying the area of its base into its perpendicular altitude; consequently, if h denote the perpendicular height of the column, we have

$$.0490875\,h = 30.72;$$

therefore, by division, we shall obtain

$$h = \frac{30.72}{.0490875} = 625.819 \text{ inches.}$$

153. The solution which we have here given, applies to the particular example preceding, in which the data are assigned; but in order to accommodate the theory to every case, it becomes necessary to draw up the solution in general terms; for which purpose, we must recur to equation (114), where the weight of the equilibrating column has already been found; then, according to the above analogy, we have

$$62\tfrac{1}{2} : 1728 :: \frac{d^2 w}{\text{D}^2} : s,$$

where s denotes the solidity of the column.

If in the above analogy, we make the product of the mean terms equal to the product of the extremes, we shall have

$$62.5s = \frac{1728\,d^2 w}{\text{D}^2},$$

and from this, by division, we get

$$s = \frac{27.648\,d^2 w}{\text{D}^2}. \qquad (115).$$

Therefore, if the solidity of the equilibrating column be divided by the area of its base, viz. the quantity $.7854d^2$, the quotient will furnish the perpendicular altitude; hence we have

$$h = \frac{35.2024\,w}{\text{D}^2}. \qquad (116).$$

154. From this it appears, that in order to determine the altitude of the equilibrating column, it is not necessary that its diameter should be known, for the equation is wholly independent of that element, the diameter of the bellows, and the weight upon the moveable board only, entering into its composition. The following practical rule will therefore determine the altitude of the column by which the equilibrium is maintained.

Rule. *Divide* 35.2024 *times the load to be sustained upon the moveable board, by the square of the diameter of the bellows, and the quotient will be the altitude of the equilibrating column.*

We shall determine the perpendicular altitude by this rule, on the supposition that the diameter of the bellows and the weight upon the moving plane, are the same as in the foregoing example; therefore we have

$$h = \frac{35.2024 \times 5760}{324} = 625.819 \text{ inches.}$$

The equation (114) for the weight of the equilibrating column, was deduced from the equation (113), by simple division only, without the enunciation of any problem; but in order to render the subject a little more systematic, we shall determine the other elements of the general equation, severally from the resolution of their respective and appropriate problems.

PROBLEM XXII.

155. In a hydrostatical bellows of a cylindrical form, there are given, the diameters of the bellows and of the equilibrating tube, together with the weight of the fluid by which the equilibrium is maintained :—

It is required to determine the weight upon the moveable plane, at the instant when the equilibrium obtains.

Let both sides of the general equation (113), be divided by d^2 the square of the diameter of the balancing tube, and we shall obtain

$$w = \frac{D^2 w'}{d^2}. \qquad (117).$$

And this equation affords the following practical rule.

Rule. *Multiply the weight of the equilibrating fluid, by the square of the diameter of the bellows, and divide the product by the square of the diameter of the tube, for the weight upon the moveable plane.*

Example. The diameter of a cylindrical bellows is 24 inches, the diameter of the balancing tube is one fourth of an inch, and the weight of the fluid in the tube is 2½ lbs.; what weight will this counterpoise on the moving board of the bellows?

Here, by proceeding as directed in the rule, we obtain

$$w = \frac{576 \times 2.5}{.0625} = 23040 \text{ lbs.}$$

This is something more than 10 tons and a quarter, which is manifestly a great load to be suspended by 2½ lbs.; but the altitude of the suspending column must be proportionably great, which circumstance, without the aid of some artificial force, would render the instrument very inconvenient for any practical purpose; it was, no doubt, by viewing the matter in this light, that Mr. Bramah, senior, was led to apply the forcing pump, and thereby to produce that very powerful engine, which formed the subject of our last article.

PROBLEM XXIII.

156. In a hydrostatical bellows of a circular form, there are given, the diameter of the bellows, the load suspended, and the weight of the suspending fluid:—

It is required to determine the diameter of the equilibrating tube, so that the instrument may be just in a state of equilibrium.

Let both sides of the general equation (113), be divided by w the weight upon the bellows, and we shall obtain

$$d^2 = \frac{D^2 w'}{w},$$

and from this, by extracting the square root, we get

$$d = \sqrt{\frac{D^2 w'}{w}}.^* \qquad (118).$$

And the practical rule which this equation supplies, may be expressed in words at length in the following manner.

RULE. *Multiply the square of the diameter of the bellows, by the weight of the fluid which maintains the equilibrium, and divide the product by the weight upon the bellows, then, the square root of the quotient will be the diameter of the equilibrating tube.*

---

* This equation for the diameter of the tube may be otherwise expressed; thus

$$d = D\sqrt{\frac{w'}{w}}.$$

Example. The diameter of the bellows or cylindrical vessel, is 24 inches, the weight of the suspending fluid is 2 lbs., and the weight suspended on the bellows 8000 lbs.; what is the diameter of the tube?

Performing according to the rule, we have

$$d^2 = \frac{24 \times 24 \times 2}{8000} = 0.144,$$

and from this, by extracting the square root, we obtain

$$d = \sqrt{0.144} = .38 \text{ of an inch.}$$

PROBLEM XXIV.

157. In a hydrostatic bellows of a cylindrical form, the diameter of the tube, the weight suspended, and the weight of the suspending fluid, are given:—

It is required to determine the diameter of the bellows, so that the whole may be in a state of equilibrium.

Let both sides of the general equation (113), be divided by w' the weight of the suspending fluid, and we shall have

$$D^2 = \frac{d^2 w}{w'}.$$

from which, by extracting the square root, we get

$$D = \sqrt{\frac{d^2 w}{w'}}.\text{*} \qquad (119).$$

And from this equation, we obtain the following practical rule.

Rule. *Multiply the square of the diameter of the suspending tube, by the weight suspended, and divide the product by the weight of the fluid which maintains the equilibrium; then, the square root of the quotient will be the diameter of the cylinder sought.*

Example. The diameter of the suspending tube in a cylindrical hydrostatic bellows, is half an inch, the weight of the suspending fluid is 2 lbs., and the weight suspended on the bellows board is 12000 lbs.; what is the diameter of the bellows?

Here, by proceeding as directed in the foregoing rule, we get

$$D^2 = \frac{.5 \times .5 \times 12000}{2} = 1500,$$

---

* This equation for the diameter of the bellows may be otherwise expressed; thus:

$$D = d\sqrt{\frac{w}{w'}}.$$

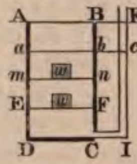
and by extracting the square root, we have

$$D = \sqrt{1500} = 38.73 \text{ inches.}$$

158. The foregoing problems and rules, unfold every particular respecting the calculation of the hydrostatic bellows, and from them we may infer, that in the case of an equilibrium, if more fluid be added :—

> *It will ascend equally in the suspending tube, and in the cylindrical vessel composing the bellows, whatever may be their relative magnitudes.*

The demonstration of this is very simple, for let ABCD be a vertical section, passing along the axis of the cylindrical vessel, and also along the axis of the suspending tube KI; and suppose that F and c are the points to which the fluid rises in the vessel and the tube, when the bellows is in a state of equilibrium.



Take Ic equal to Da, and through the points a and c let a horizontal plane be drawn, intersecting the vertical plane ABCD in the line ab; then it is manifest, that the weight w in the position EF, is equivalent to the weight of the fluid column abFE. Let more fluid be poured into the tube at K; the equilibrium will then be destroyed, and the weight w will ascend, until by discontinuing the supply, the equilibrium is restored, and the fluid in the vessel and the tube becomes again quiescent at the points n and K.

Take IK equal to DA, and through the points A and K, let a horizontal plane be drawn, cutting the vertical plane ABCD in the line AB; then as before, the weight w in the position mn, is equivalent to the weight of the fluid cylinder, of which ABnm is a vertical section.

Now, the weight w is not altered in consequence of the change of position from EF to mn; therefore, because EF is equal to mn, it follows, that Ea is equal to mA; consequently, by taking away the common space ma, the remainders Em and aA are equal to one another; but by reason of the parallel lines ac and AK, the spaces aA and cK are equal to one another; therefore cK is equal to Em.

From the principle here demonstrated, the resolution of the following problem may readily be derived.

PROBLEM XXV.

159. If a hydrostatic bellows of a cylindrical form, have a given quantity of fluid poured into the equilibrating or suspending tube:—

It is required to determine through what space the weight on the moving board will ascend in consequence of the supply.

Before we proceed to the resolution of this problem, it may be proper, as in the foregoing cases, to exhibit an appropriate notation; for which purpose,

Put $D = AB$ or DC, the diameter of the cylindrical vessel or bellows,
$d =$ the diameter of the equilibrating or suspending tube,
$q =$ the quantity of fluid poured into the tube, and
$x = Em$, the space through which the weight ascends by reason of the supply.

Then, according to the principles of mensuration, the area of a transverse section of the cylindrical vessel or bellows, is

$$a = .7854\,D^2,$$

and the area of the corresponding section of the tube, is

$$a' = .7854\,d^2,$$

where the symbols a and a' denote the respective areas.

But by the property demonstrated above, the fluid rises equally in the bellows and in the tube; therefore, the quantity of fluid which flows into the bellows in consequence of the supply, is

$$s = .7854\,d^2 x,$$

and the quantity which remains in the tube, is

$$s' = .7854\,d^2 x,$$

where the symbols s and s' denote the solidities of the cylinders, whose diameters are D and d, and their common altitude x.

Now, the sum of these quantities, is manifestly equal to the quantity of fluid poured into the tube; hence we have

$$q = .7854\,x(D^2 + d^2),$$

and by division, we obtain

$$x = \frac{q}{.7854(D^2 + d^2)}. \qquad (120).$$

It therefore appears, that the space through which the weight ascends by reason of the supply :—

> *Is equal to the quantity of fluid which is poured into the tube, divided by the sum of the areas of a cross section of the tube and the cylindrical vessel or bellows.*

The practical rule, or method of applying the equation, may therefore be expressed in words at length in the following manner.

> Rule. *Divide the quantity of fluid which is poured into the tube, by .7854 times the sum of the squares of the diameters, and the quotient will give the quantity of ascent, or the space through which the weight is raised in consequence of the supply.*

Example. The diameter of a cylindrical vessel is 20 inches, and that of the suspending tube is one inch ; now, suppose that an incompressible fluid is poured into the tube, until its weight sustain in equilibrio, a load of 8760 lbs. upon the moveable bellows board; then, how much higher will the load be raised, when 150 cubic inches of the fluid are superadded ?

Here then, we have given $D = 20$ inches, $d = 1$ inch, and $q = 150$ cubic inches ; consequently, by the rule, we obtain

$$x = \frac{150}{.7854(400+1)} = .476 \text{ of an inch.}$$

From this it appears, that if a machine of the given dimensions be brought into a state of equilibrium, the addition of 150 cubic inches will raise the load .476, or very nearly half an inch higher ; in which case the equilibrium will still obtain, for the altitude of the fluid in the suspending tube, is increased exactly as much as the load has been raised, while the magnitude of the load, and consequently, the height of the equilibrating column, remain the same.

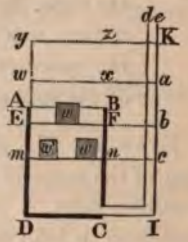
3. THEORY OF CONSTRUCTION AND SCIENTIFIC DESCRIPTION OF THE HYDROSTATIC WEIGHING MACHINE.

160. The preceding principles may also be applied to the construction of a very simple and convenient weighing machine ; for, if into the side of an open cylindrical or other vessel, a bent tube be inserted, and if on the surface of the fluid, a moveable cover exactly

fitting the vessel be placed with a weight upon it, and the tube graduated:—

> *Then, any additional weight placed upon the cover, may be determined by knowing the height to which the fluid rises in the tube;* and conversely:—
>
> *If the additional weight be known, the height to which the fluid rises in the tube may be found.*

Let ABCD represent a vertical section of a cylindrical vessel, or of any other vessel, whose sides are perpendicular to the horizon; and let KIC be the corresponding section of the equilibrating tube.



Let both the vessel and the communicating tube be open at the upper parts AB and *de*, and conceive the vessel to be filled with fluid to the line EF or altitude DE; then, on the surface of the fluid at EF, let there be placed a moveable cover exactly fitting the vessel, so that the whole may be water-tight.

Produce EF to *b*, then is the point *b* at the same level in the tube IK, as the surface of the fluid in the vessel whose level is EF: upon the cover EF let the weight w be placed, and suppose a to be the point in the tube, to which the fluid will rise by the action of the cover, together with the weight w which is placed upon it; in this case, the machine is in a state of equilibrium.

If some additional weight w' be placed upon the cover, then the original equilibrium will be destroyed, and can only be restored, by the fluid ascending in the tube to a sufficient height to balance the additional weight.

Put $D = AB$ or DC, the diameter of the cylindrical vessel, of which ABCD is a section,

$d = de$, the diameter of the communicating tube KIC,

$h = ba$, the height of the original equilibrating column,

$w =$ the weight supported by the column ba,

$w' =$ the additional weight, whose quantity is required,

$h' = aK$, the increased altitude of the supporting column,

$\delta = Em$, the descent of the cover occasioned by the additional load w', and

$s =$ the specific gravity of the fluid.

Then it is manifest, that when the equilibrium originally obtains; that is, when the surface of the fluid in the tube is at a, and that in the vessel at EF, the pressure of the fluid in the tube exerted at b, is

$$p = .7854 d^2 h s,$$

where the symbol p denotes the pressure at b;

but this is manifestly in equilibrio with the pressure of the column $w\mathrm{EF}x$, or the weight w; consequently, we have

$$.7854 d^2 h s : w :: .7854 d^2 : .7854 \mathrm{D}^2;$$

or by suppressing the common factors, we obtain

$$h s : w :: 1 : .7854 \mathrm{D}^2;$$

and by equating the products of the extremes and means, it is

$$w = .7854 h s \mathrm{D}^2. \qquad (121).$$

Again, by means of the additional weight w', whose magnitude is required, the cover EF is supposed to descend to the position mn; while, in order to regain the equilibrium, the fluid rises in the tube as far as the point K, in which case, the altitude of the equilibrating column cK becomes $(h + h' + \delta)$ and consequently, the pressure at c, is

$$p' = .7854 d^2 s (h + h' + \delta),$$

and this is in equilibrio with the pressure of the column $ymnz$, or the weight $(w + w')$; consequently, we have

$$.7854 d^2 s (h + h' + \delta) : w + w' :: .7854 d^2 : .7854 \mathrm{D}^2;$$

or by suppressing the common factors, we have

$$s (h + h' + \delta) : w + w' :: 1 : .7854 \mathrm{D}^2;$$

therefore, by equating the products of the extremes and means, we get

$$w + w' = .7854 \mathrm{D}^2 s (h + h' + \delta). \qquad (122).$$

But we have seen above, equation (121), that $w = .7854 h s \mathrm{D}^2$; consequently, by substituting and separating the terms, we obtain

$$w' = .7854 \mathrm{D}^2 s (h' + \delta). \qquad (123).$$

Now, it is manifest, that the descent of the cover in the vessel, and the rise of the fluid in the tube, must be to one another, inversely as the squares of the respective diameters; therefore, we have

$$\delta \mathrm{D}^2 = h' d^2,$$

or by division, we get

$$\delta = \frac{h' d^2}{\mathrm{D}^2},$$

and finally, by substitution, we obtain

$$w' = .7854 h' s (\mathrm{D}^2 + d^2). \qquad (124).$$

161. If the fluid be water, whose specific gravity is represented by unity, the equation becomes somewhat simpler; for in that case, we have

$$w' = .7854\,h'\,(\text{D}^2 + d^2). \qquad (125).$$

From this equation the magnitude of the additional weight, or the measure by which it is expressed, can very easily be ascertained; and the practical rule by which it is discovered, is as follows.

> Rule. *Multiply the sum of the squares of the diameters, by .7854 times the rise of the fluid in the tube, or the elevation above the first level, and the product will express the magnitude of the additional weight.*

Example. The diameter of a cylindrical vessel is 16 inches, and that of the communicating tube one inch; now, supposing the machine in the first instance, to be in a state of equilibrium, and that by the addition of a certain weight on the moveable cover, the water in the tube rises 6 inches above the original equilibrating level; how much weight has been added?

By proceeding according to the rule, we have

$$\text{D}^2 + d^2 = 16^2 + 1^2 = 256 + 1 = 257,$$

and by multiplication, we obtain

$$w' = .7854 \times 6 \times 257 = 1211.0868 \text{ avoirdupois lbs.*}$$

162. If the additional weight, by which the water is made to rise in the tube be given, the distance above the first level to which it will rise, can easily be found; for let both sides of the equation (125), be divided by the quantity $.7854\,(\text{D}^2 + d^2)$, and we shall obtain

$$h' = \frac{w'}{.7854\,(\text{D}^2 + d^2)}.$$

And from this equation, we deduce the following rule.

> Rule. *Divide the additional weight, by the sum of the areas of the moveable cover and the cross section of the communicating tube, and the quotient will give the height to which the fluid will rise above the first level.*

---

* It is manifest from the form of the equation which supplies the rule, that without paying particular attention to the nature of the load which produces the equilibrium in the first place, the value of w' is ambiguous, and may be read in ounces, lbs., cwts., or tons; and indeed, in any denomination of weight whatever; but it must always be read in the same name as that by which the equilibrium is produced.

EXAMPLE. The diameter of the moveable cover is 16 inches, and that of the communicating tube one inch; then, supposing that the machine in the first instance is brought to a state of equilibrium, and that a load of 1211 lbs. is applied on the cover, in addition to that which produces the equipoise; to what height above the first level will the water ascend in the communicating tube?

Proceeding according to the rule, we obtain

$$.7854\,(\mathrm{D}^2+d^2)=.7854\,(16^2+1^2)=201.8478 \text{ divisor};$$

consequently, by division it is

$$h'=\frac{1211}{201.8478}=6 \text{ inches nearly.}$$

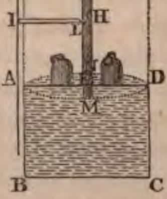
And exactly after the manner of these two examples, may any other case be calculated; but in applying the principles to the determination of weights, mercury ought to be employed in preference to water, as it exerts an equal influence in less space, and besides, it is not subject to a change of density by putrefaction and the like.

4. EXPERIMENTS ILLUSTRATING THE QUAQUAVERSUS PRESSURE OF INCOMPRESSIBLE FLUIDS.

Before we conclude our inquiries on fluid pressure, it may be both interesting and instructive to the readers of this work, to describe a few select experiments, by which the equal distribution of pressure, among the particles of an incompressible fluid is beautifully and rigorously demonstrated, and its equal propagation in all directions, placed beyond the possibility of the smallest doubt.

Allied to the preceding subject, is the following, by which is exhibited a very surprising effect of the equilibrium of incompressible fluids, but which, for the sake of convenience, we shall suppose to be water, since that is more easily obtained in small or large quantities, than any other fluid whatever.

EXPERIMENT 1. Let ABCD represent an upright section of a square or cylindrical vessel, closed at top with a cover of which AD is a section; make a hole in the top at E, and fix a tube FE therein of any convenient diameter at pleasure, but small in comparison of the diameter of the vessel. Let the tube be closely fixed in the cover with pitch, or some other glutinous matter, so as to be rendered air and water-tight all round the orifice, and suppose its length or height to be twelve or fifteen inches according to circumstances; then, fill the vessel with water by some holes made



in the top and afterwards stopped up, or it may be filled through the tube alone.

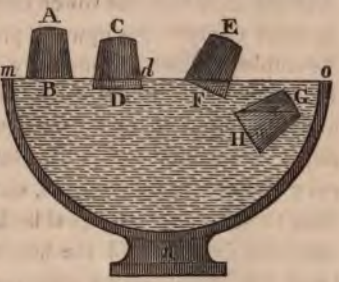
Now, if a load of about seven or eight hundred lbs., be laid upon the cover of the vessel, it will be depressed into a concavity represented by the dotted line A M D, the displaced water ascending in the tube, in proportion as the cover is bent by the pressure of the superincumbent load; but if we pour water into the tube F E, the cover of the vessel, together with its incumbent load, will not only be raised to the original situation, but will even assume a convex form, as represented by the dotted line A N D, rising in the middle as much above the point E, as it was formerly depressed below it, the quantity of elevation being measured by the index or ruler I L, which is fixed in an adjoining support in such a manner, as to remain immoveable, the point H which is marked on the tube, ascending or descending with the cover of the vessel.

If the tube be increased in length and more water added, it will be found that the cover of the vessel, together with its load, will rise higher and higher until a rupture takes place by overstretching the fibres of the material; this however, is a case not admitted in the experiment, and consequently, we may conclude, that the small column of water in the tube :—

> *Exerts the same force in raising the cover of the vessel, together with the load upon it, as if the tube and the vessel were of equal diameters, and the incumbent column equal to* A G H D, *instead of that contained in the tube.*

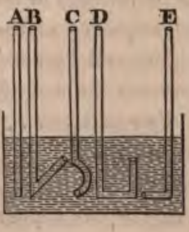
Now, this is precisely the property of the weighing machine formerly exemplified; for the water in the small tube F E, will raise the cover of the vessel, (supposing it to be moveable and water-tight), together with its load, even although it were a thousand times greater: this is manifest, because the velocity with which the water descends in the tube, is to the velocity with which it ascends in the vessel, as the area of a section of the vessel, is to the area of a corresponding section of the tube; for instance, if the vessel is 30 inches in diameter, while the supplying tube is only one; then, we know by the principles of mensuration, that the area of the top of the vessel, is to that of a section of the tube, in the ratio of 900 to unity; consequently, when the water in the tube has descended one inch, the top of the vessel, and the load upon it, has ascended by a one nine-hundredth part of an inch; therefore, if the water in the tube weighs one lb., it will be in equilibrio with 900 lbs. in the vessel, or which is the same thing, one lb. of water in the tube, will suspend the top of the vessel, together with the load upon it, supposing them to weigh conjointly 900 lbs.

EXPERIMENT 2. Let *mno* represent a vertical section of a spherical vessel filled with water, or some other incompressible and non-elastic fluid, and let AB be a common tumbler glass, held vertically with its mouth exactly in contact with the fluid's surface; then it is manifest, that in this state the glass is completely filled with air of the natural density; that is, with air of the same density as atmospheric air on the surface of the earth.



If the tumbler be still held in a vertical position, but a little depressed below the surface of the fluid, as represented by CD, then it is obvious, that a small quantity of the fluid has entered, and the rest of the glass is filled with air in a state of slight condensation, corresponding to the pressure of the superincumbent column of water represented by D*d*. And moreover, if the glass be still farther depressed, the fluid will ascend higher and higher, and the air will be compressed into a less and less space.

Again, if the glass be inclined in any degree from the vertical position, as represented by EF and GH, taking care to have its mouth wholly immersed in the water, then it is evident, that the greater the degree of inclination, the greater is the quantity of fluid which enters, and the greater also is the condensation of the included air; but when the quantity of fluid which enters the glass is the same, both in the vertical and the inclined position, the density of the air is also the same, being compressed by the same force; consequently, the water or fluid in which the glass is placed, exerts the same pressure in whatever direction it is propagated. One sees this experiment verified daily by empty casks having only one end, thrown into water.

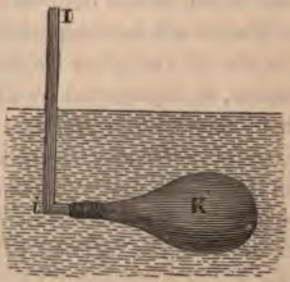


EXPERIMENT 3. If the several tubes A, B, C, D and E, bent at various angles, be inserted in an empty vessel, or if they be held in the hand, and mercury be introduced at their lower extremities, in such a manner, as to come close to the orifices; then let water be poured into the vessel, and it will be seen, that during the time of its filling, the mercury is pressed gradually from the

lower towards the higher extremities of the tubes, which are supposed to rise to a height considerably above the surface of the water.

Now, since the lower extremities of the tubes may be conceived to point in every possible direction, it follows, that the pressure of the superincumbent fluid is also propagated in every direction. But when it is required that the lower orifice should point directly downwards, in order to show the upward pressure of fluids, a straight tube must be employed, and the mercury which is introduced must be kept in by the finger, until the height of the water above the lower surface, is about fourteen times the height of the mercurial column; for if the finger be removed before the water has attained that height, the mercury will fall out of the tube, since its weight is fourteen times greater than the weight of an equal bulk of water. If the finger be continued upon the orifice, until the height of the water be equal to fourteen times the height of the mercury, then, on removing the finger, and pouring in more water, the mercury will be seen to ascend in the tube, and will continue to rise higher and higher, according to the quantity of water poured in, thereby showing the upward pressure of the water.

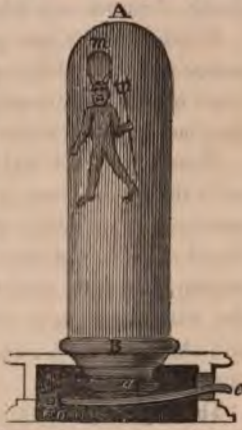
EXPERIMENT 4. The pressure of fluids at diifferent points of their depths, may be very simply illustrated in the following manner: let K be a bag of leather, or some other tough and flexible material, filled with mercury, and attached to the extremity of a glass tube *i*I, in such a manner, that the mercury may just enter the tube when the bag is held in air.



Then, if the bag be immersed in water, it is manifest that the pressure of the fluid will cause it to collapse, and the mercury will ascend in the tube to a certain height, corresponding to the pressure exerted by the water, at the depth where the bag is placed. If the bag continue to be lowered in the water, it will become more and more collapsed in consequence of the increased pressure, and the mercury will ascend higher and higher in the tube, and the heights to which it rises, will indicate the magnitude of pressure at different depths.

EXPERIMENT 5. There is a very simple and amusing experiment, by which the propagation of pressure through fluids is illustrated, called the "*Cartesian Devil*," from M. Descartes, the celebrated French philosopher, by whom it was discovered; it is as follows.

Let the little figure in the inverted jar A B represent the "*Cartesian Devil*," surmounted by a bag-like crown of great size in proportion to his body, filled with some very light substance, such as air, and we shall therefore suppose that air is the body which it contains. The imp himself must be constructed of glass or enamel, so as to possess the same specific gravity as water, and therefore to remain suspended in the fluid.



At the bottom of the vessel or jar, is placed a diaphragm or bladder, that can be pressed upwards by applying the finger to the extremity of a lever *eo*, moving round *o* as its fulcrum or centre of motion. The pressure applied at *a* is communicated through the water to the bag of air at *m*, which is thus compressed, and consequently, the specific gravity of the figure is increased, by which it sinks to the bottom of the jar.

By removing the pressure on the diaphragm at *a*, the figure will again ascend, so that it may be made to oscillate, or rise upwards and sink downwards alternately, and to dance about in the jar, without any visible cause for its movements.

Other figures, such as fishes made of glass, are sometimes employed in this experiment, but the principle is nevertheless the same, and when a common jar is used, the pressure is applied to the upper surface of it, as at A.

EXPERIMENT 6. The pressure of fluids at very great depths, is beautifully illustrated by an experiment which has often been made at sea, where the water is sufficiently deep to admit of the principle being accurately put to the trial.

The experiment is this: an empty bottle well corked is made to descend to a great depth, on which the pressure of the fluid becomes so great as to drive in the cork, and the bottle when brought up is always filled with water. Several methods have been employed to prevent the cork from being driven inwards, but although this has been effected, yet the bottle on being brought to the surface, is constantly filled with the fluid in which it has been sunk.

The following experiments of this sort, are detailed by Mr. Campbell, the author of "*Travels in the South of Africa*," published at London in the year 1815; the experiments were tried on his voyage homewards from the Cape of Good Hope. He drove very tightly into an empty bottle, a cork of such a large size, that one half of it remained above the neck: a cord was then tied round the cork and fastened to the neck of the bottle, and a coating of pitch was put over the whole.

When the bottle was let down to the depth of about fifty fathoms, he perceived by the additional weight, that it had instantly filled; and on drawing it up, the cork was found in the inside of the bottle, which of course was filled with water.

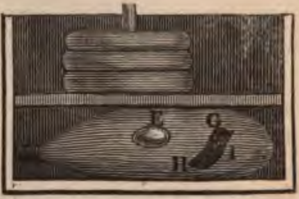
Another bottle was prepared in a similar manner; but in order to secure the cork, and to prevent it from being pressed within the bottle, a sail needle was passed through it, so as just to rest on the margin of the glass, and the whole was carefully covered with a coating of pitch.

When the bottle had descended to the depth of about fifty fathoms, as in the former case, it was again perceived to have been filled with water; and on bringing it to the surface, the cork and needle were found in the same position, and no part of the pitch appeared to be broken, although the bottle was completely filled with water. Here the water must have insinuated itself through the pores of the pitch and the cork, and not as the experimentalist supposes, through the pores of the glass.

The equality of fluid pressure in every direction, is very easily demonstrated in the following manner.

Experiment 7. If a piece of very soft wax, as G H I, and the egg E, be placed in a bladder, or some other flexible vessel filled with water, and if the bladder be put into a brass box, and a moveable cover laid upon the bladder so as to be wholly supported by it.

Then, if one hundred, or one hundred and fifty lbs. be laid upon this cover, so as to press upon the bladder and its contained fluid; this enormous force, although propagated throughout the fluid, and acting upon the soft wax and the egg, will produce no effect, the wax will not change its form, and the egg will not be broken. And in like manner, if a living fish should be put into the cylinder of a hydrostatic press, when under a very high degree of pressure, it will not suffer the least inconvenience; from which it is obvious that every particle of the fluid is equally pressed, and presses equally in all directions.



Numerous other examples might be adduced for proving the same thing, but since the principle is manifest, it is needless to dwell longer on the subject.

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CHAPTER XI.

OF THE EQUILIBRIUM OF FLOATATION.

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It is observed of Archimedes, by his philosophical biographer, that "although we might labour long without success in endeavouring to demonstrate from our own invention, the truth of his propositions; yet so smooth and so direct is the way by which he leads us, that when we have once travelled it, we fancy that we could readily have found it without assistance, since either his natural genius, or his indefatigable application, has given to every thing that he attempted the appearance of having been performed with ease."

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CHAPTER VII.

OF PRESSURE AS IT UNFOLDS ITSELF IN THE ACTION OF FLUIDS OF VARIABLE DENSITY, OR SUCH AS HAVE THEIR DENSITIES REGULATED BY CERTAIN CONDITIONS DEPENDENT UPON PARTICULAR LAWS, WHETHER EXCITED BY MOTION, BY MIXTURE, OR BY CHANGE OF TEMPERATURE.

In the former part of this treatise, we have displayed the nature of pressure as it occurs in the action of non-elastic fluids of uniform density, and in addition, we have investigated the theory and exemplified the application of the *Hydrostatic Press*, the *Hydrostatic Bellows*, and the *Hydrostatic Balance* or weighing machine; instruments whose operations depend upon the *quaqua-versus* principle of non-elastic and incompressible fluids:—We come therefore in the next place, to consider pressure as it unfolds itself in the action of fluids of variable density, or such as have their densities regulated by certain conditions, dependent upon particular laws, whether excited by motion, by mixture, or by change of temperature.

In mechanical science density is used as a term of comparison, expressing the proportion of the number of equal moleculæ in the same bulk of another body; density, therefore, is directly as the quantity of matter; and inversely as the magnitude of the body.

We cannot by means of our senses discover the figure and magnitude of the elementary particles of matter. Mechanical inventions have wonderfully magnified objects invisible to the unassisted eye; but no microscopical assistance has yet enabled us to assume that we have seen an elementary particle of matter. A number of elementary particles uniting by the power of cohesion form greater particles, and these again uniting, by the same power, form still greater; and we may consider the aggregate of many such formations to become at length an atom of a sensible bulk. All bodies seem to be composed of these derivative corpuscles, which, formed of more or fewer of these repeated unions, compose bodies more or less dense. These derivative corpuscles are sometimes similar, as the coloured rays of a beam of light separated by the prism; mercury, when squeezed through the pores of leather, or raised in fume and received upon clean glass, which exhibits globules similar and undistinguishable. In short, every mass of matter is divisible into particles, which we designate by the Greek term *atom*, or that which is so exceedingly minute that it cannot be further cut or divided, and which therefore, as far as sense is concerned, is the ultimate resisting particle. It must be obvious, that the density or quantity of atoms which exist in a given space is very different in different substances. Hence, if it be asked why bodies are called dense? the answer is, Because they contain more atoms than others of the same size. There are more atoms in a cubic inch of lead than in a cubic inch of cork: the former is forty times heavier than the latter. A cubic foot of rain water weighs 62½ lbs.; but an equal volume of mercury, which is fourteen times heavier than water, weighs $(62\frac{1}{2}\times 14)=875$ lbs.

Density must depend on three circumstances, to which we should carefully attend in all our disquisitions: first, the size or weight of the individual atom; secondly, on porosity, or the arrangement of the atoms by cohesion, or mechanical and physical arrangement; thirdly, the proximity of the atoms determined by the substance of which they are constituent particles, possessing tenacity and incompressibility. Thus, heat dilates some bodies and contracts others. A pound of tin and a pound of copper melted together form bronze; but this new mass occupies less space by one fifteenth than the two masses did when separate; proving that the atoms of the one are partially received into what were empty spaces of the other. In other words, the affinity of cohesion is one fifteenth greater in the bronze than in the tin and copper separately. Two pounds of brine are made out of a pound of salt and a pound of water; but the mass is of less bulk than the aggregate of the ingredients apart.

Water, we have seen, resists compression very powerfully, but at the depth of 1000 fathoms yielding a very small part of its bulk at the surface, shows the particles not to be in contact, and that the fluid may acquire density in proportion to its depth. Wood swims in water, because the water has more atoms in the same bulk than the wood, and therefore more weight or central force than the wood; consequently, the water falls first and leaves the wood behind; in other words, the wood floats upon the water—the wood is borne on the surface of the water with a force exactly proportional to the difference between its weight and that of an equal bulk of water. The pressures which the fluid exerts in supporting the wood are together equivalent to a force directed upwards through the centre of gravity of the fluid displaced, and equal to the weight of a quantity of the fluid so displaced by the immersed part of the body. But it is not necessary here to dwell further on this topic, the density of water. We therefore pass on to another character it possesses, viz. gravity or weight: and it is, in fact, by comparing the weight of a body with the force which holds it up in the fluid, that the comparative weights or specific gravities are found, as of metals compared with water, and of admixtures of metals for the purpose of ascertaining at once the proportion of each in the compound mass.

Water is the common standard with which all other substances are compared, whose weight we would fix and record in tables of specific gravities. When we say, therefore, that gold is of the specific gravity of 19, and copper of 9, and cork of one seventh, we mean that these substances are just so much heavier or lighter than their bulk of pure water in its densest form, viz. at the temperature of 40 degrees of Fahrenheit's thermometer. It appears, therefore, that the terms density and specific gravity express the same thing under different aspects; the former being more accurately restrained to the greater or less vicinity of particles, the latter to a greater or less weight in a given volume; hence, as weight depends upon the closeness of the particles, the density varies as the specific gravity, and the terms may in most cases be indiscriminately used.

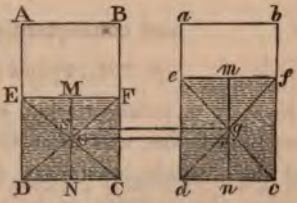
The specific gravities of fluids are usually considered without any regard to the empty spaces between the particles, though if the particles of fluids are spherical, the vacuities make at least one fourth of the whole bulk. But it is sufficient that we know precisely in what sense the specific gravities of fluids are understood.

PROBLEM XXV.

163. A cylindrical vessel whose sides are perpendicular to the horizon, has a certain quantity of fluid in it; which fluid, by reason of a sudden change of temperature, has its magnitude or bulk increased by a certain part of itself:—

It is therefore required to determine what will be the alteration of pressure on the sides and bottom of the vessel.

Let ABCD, and $abcd$ respectively, represent vertical sections of the cylindrical vessel, of which the sides are perpendicular and the base parallel to the horizon; then in the first instance, let EF be the height to which the vessel is filled, and ef the height to which the fluid rises, by reason of the change that takes place in the temperature.



Draw the diagonals EC, FD and ec, fd intersecting respectively in the points G and g, and through the points G, g, draw the vertical lines MN and mn; then are MG and mg, the respective depths of the centres of gravity of the cylindric surfaces, in contact with the fluid before and after the expansion, and MN, mn, are the depths of the centres of gravity of the bases or bottoms DC and dc.

Through the points G and g, draw the straight lines Gr and gs, parallel to the horizon and to one another; then is Gs or rg the height which the centre of gravity of the cylindric surface is elevated, by reason of the expansion of the fluid.

Put $d =$ DC or dc, the diameter of the cylindric vessel,
$h =$ MN, the height to which the vessel is originally filled,
$h' = mn$, the height at which the fluid stands in the vessel after expansion,

Put $P =$ the pressure on the bottom DC, by the fluid in its original state,
$p =$ the corresponding pressure on the cylindric surface,
$P' =$ the pressure on the bottom dc, by the fluid after expansion,
$p' =$ the corresponding pressure on the cylindric surface,
$s =$ the specific gravity of the fluid before expansion,
$s' =$ the specific gravity of the fluid after expansion,
$a =$ the area of the base or bottom of the vessel in both cases;
ϕ and ϕ' the cylindric surfaces, and
$\frac{1}{n} =$ the part of its bulk by which the fluid is increased.

Then, since d denotes the diameter of the bottom, the area according to the principles of mensuration, becomes

$$a = .7854\,d^2,$$

and the pressure exerted by the fluid in its original state, is

$$P = .7854d^2hs. \qquad (126).$$

Again, according to the principles of mensuration, the cylindric surface in contact with the fluid before expansion, is

$$\phi = 3.1416dh,$$

and consequently, the pressure upon it, is

$$p = 3.1416dh \times \tfrac{1}{2}h \times s = 1.5708dh^2s. \qquad (127).$$

Now, it is manifest, that since the diameter of the vessel is the same both before and after the expansion of the fluid, the capacity and the altitude must vary directly as each other; consequently, because the capacity or bulk is increased by $\frac{1}{n}$th part of itself, it follows, that the altitude is increased in the same proportion; therefore we have

$$h' = h\left(\frac{n+1}{n}\right);$$

but when the weight of the fluid remains the same, the density, and consequently the specific gravity, *varies inversely as the magnitude.*

The specific gravity of the fluid, after it has expanded by reason of an increase of temperature, is therefore,

$$\frac{h(n+1)}{n} : h :: s : s' = \frac{ns}{n+1};$$

hence, the pressure on the bottom of the vessel, after the fluid has increased by expansion, becomes

$$P' = .7854d^2h's'; \text{ that is,}$$

$$P' = .7854d^2 \times h\left(\frac{n+1}{n}\right) \times \frac{ns}{n+1} = .7854d^2hs. \quad (128).$$

The cylindric surface in contact with the fluid after expansion, may be expressed as follows, viz.

$$\phi'\left(=\phi\frac{n+1}{n}\right);$$

but it has been shown above, that

$$\phi = 3.1416dh;$$

therefore, by substitution, we obtain

$$\phi' = 3.1416dh\left(\frac{n+1}{n}\right),$$

and consequently, the pressure becomes

$$p' = 3.1416dh\left(\frac{n+1}{n}\right) \times \tfrac{1}{2}h\left(\frac{n+1}{n}\right) \times \frac{ns}{n+1} = 1.5708dh^2s\left(\frac{n+1}{n}\right) (129).$$

If therefore, the equations (126) and (128) be compared with one another, it will be found that the pressure is the same, and equal to the weight of the fluid in both cases; but if the equations (127) and (129) be compared, the pressure in the one case, is to that in the other, in the ratio of $n : n+1$; that is,

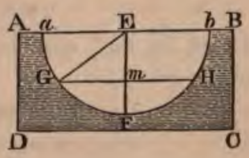
$$p : p' :: n : n+1.$$

PROBLEM XXVI.

164. A semi-circular plane is vertically immersed in a fluid whose density increases as the depth, and in such a manner, that the horizontal diameter coincides with the upper surface of the fluid:—

> *It is required to determine, on which chord parallel to the horizon, the pressure is a maximum, or greater than the pressure on any other chord.*

Let ABCD represent a vertical section of a mass of fluid, of which AB is the surface, and whose density varies directly as its depth; and let $aGFHB$ be the semi-circular plane immersed in it, in such a manner, that the horizontal diameter ab, coincides with AB the upper surface.



Let m be the point in the vertical

radius EF, through which the chord of maximum pressure is supposed to pass; draw the chord GH, and the radius EG; then, because the chord GH is parallel to the diameter *ab*, it follows, that GH is bisected in *m* by the vertical radius EF; consequently, *m* is the place of the centre of gravity of the chord GH, and E*m* is its perpendicular depth below AB, the upper surface of the fluid.

Put $r =$ EG, the radius of the semi-circular plane,
$\phi =$ GF, half the arc subtended by the required chord GH,
and $x =$ E*m*, the distance of the chord below the surface of the fluid.

Then, because by the conditions of the problem, the density of the fluid varies directly as its depth; it follows, that the pressure on the chord GH varies directly as G*m* drawn into Em^2; that is,

$$p : x^2\sqrt{r^2 - x^2},$$

where *p* denotes the pressure upon the chord; but this, by the conditions of the problem, is to be a maximum; therefore, we have

$$x^2\sqrt{r^2 - x^2} = \text{a maximum},$$

from which, by equating the fluxion with zero, we get

$$4r^2x^3\dot{x} - 6x^5\dot{x} = 0,$$

or by transposing and expunging the common factors, we obtain

$$3x^2 = 2r^2;$$

therefore, by division, we have

$$x^2 = \tfrac{2}{3}r^2,$$

and finally, by evolution, it becomes

$$x = r\sqrt{\tfrac{2}{3}}. \qquad (130).$$

The same result, however, may be otherwise determined; for by the arithmetic of sines, we have, to radius unity

$$\text{G}m = \sin.\phi, \text{ and } \text{E}m = \cos.\phi;$$

but in order to accommodate these quantities to the radius *r*, it is

$$\text{G}m = r\sin.\phi, \text{ and } \text{E}m^2 = x^2 = r^2\cos.^2\phi;$$

consequently, by multiplication, we obtain

$$\text{G}m \times \text{E}m^2 = r^3\sin.\phi\cos.^2\phi,$$

and this, by the conditions of the problem, is to be a maximum; hence we get

$$r^3\sin.\phi\cos.^2\phi = \text{a maximum},$$

which being thrown into fluxions, becomes

$$0 = r^3(\dot{\phi}\cos.^3\phi - 2\dot{\phi}\sin.^2\phi\cos.\phi);$$

therefore, by transposing and casting out the common terms, it is

$$\cos.^2\phi = 2\sin.^2\phi.$$

But according to the principles of Plane Trigonometry, we have $2\sin.^2\phi = 2 - 2\cos.^2\phi$; consequently, by substitution, the above expression becomes

$$\cos.^2\phi = 2 - 2\cos.^2\phi;$$

therefore, by transposition and division, we obtain

$$\cos.^2\phi = \tfrac{2}{3},$$

and by extracting the square root, we get

$$\cos.\phi = \sqrt{\tfrac{2}{3}};$$

finally, let both sides of this expression be multiplied by r, for the purpose of adapting it to the proper radius, and we shall have

$$x = r\cos.\phi = r\sqrt{\tfrac{2}{3}},\ \text{the same as above.}$$

165. The practical rule for reducing this equation, may be expressed in words at length in the following manner.

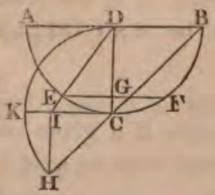
> RULE. *Multiply one third of the radius of the given semi-circular plane by the square root of* 6, *or by the constant number* 2.44947, *and the product will give the distance of the point on the vertical axis below the surface of the fluid, through which the chord of maximum pressure passes.*

166. EXAMPLE. The radius of a semi-circular plane, immersed in a fluid agreeably to the conditions of the problem, is 27 inches; at what distance below the surface of the fluid must a horizontal chord be drawn, so that the pressure which it sustains may be greater than the pressure sustained by any other chord drawn parallel to it?

By operating according to the rule, we shall obtain

$$x = 9\sqrt{6} = 9 \times 2.44947 = 22.04523 \text{ inches.}$$

167. The same example admits of a very simple and elegant geometrical construction, which may be effected in the following manner.



Let ACB be the semi-circular plane, of which the diameter AB is parallel, and the radius DC perpendicular to the horizon; draw the chord BC, and from C as a centre, with the radius CD, describe the circular arc DKH cutting BC produced in the point H.

Through the point C draw the tangent CK, and let fall the perpendicular HI meeting CK in I; join DI to intersect the arc AEC in E, and through the point E draw the chord EGF parallel to AB the diameter of the semicircle; then is EGF the chord, on which the pressure is a maximum.

That the line DG corresponds with x in the equation marked (168), may be thus demonstrated.

By reason of the parallel lines AB and KC, the angles ABH and KCH are equal to one another; but the angle ABH is manifestly equal to half a right angle or 45 degrees, therefore, the angles KCH and CHI, are each of them equal to half a right angle, and the lines CI and HI are equal, being respectively the sine and cosine of 45 degrees to the radius CH or CD.

Now, according to the principles of Plane Trigonometry, the sine and cosine of 45 degrees to the radius unity, are respectively expressed by $\frac{1}{2}\sqrt{2}$; hence we have

$$\mathrm{CI} = \frac{r}{2}\sqrt{2},$$

and by the property of the rightangled triangle, it is

$$\mathrm{DI} = \sqrt{\mathrm{CI}^2 + \mathrm{DC}^2} = r\sqrt{\tfrac{3}{2}},$$

and by similar triangles, we have

$$r\sqrt{\tfrac{3}{2}} : r :: r : \mathrm{DG} = r\sqrt{\tfrac{2}{3}}.$$

The length of the chord line EF is very easily found, for by reason of the right angled triangle EDG, of which the two sides DE and DG are known, it is

$$\mathrm{EG}^2 = \mathrm{DE}^2 - \mathrm{DG}^2;$$

but by the elements of geometry, the square of a line is equal to four times the square of its half, therefore, we have

$$\mathrm{EF}^2 = 4(\mathrm{DE}^2 - \mathrm{DG}^2);$$

hence, by extracting the square root, we get

$$\mathrm{EF} = 2\sqrt{\mathrm{DE}^2 - \mathrm{DG}^2};$$

now $\mathrm{DE}^2 = r^2$, and $\mathrm{DG}^2 = \frac{2}{3}r^2$; therefore it is

$$\mathrm{EF} = \tfrac{2}{3}r\sqrt{3}. \qquad (131).$$

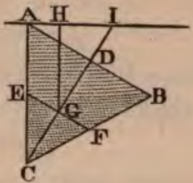
Wherefore, if we take the radius of the semi-circle equal to 27 inches, as in the preceding example, the whole length of the chord will be $18 \times 1.732 = 31.176$ inches.

PROBLEM XXVII.

168. If a given conical vessel be filled with fluid, and supported with its axis inclined to the horizon at a given angle:—

It is required to determine, on what section parallel to the base the pressure is a maximum.

Let ABC be a section passing along the axis of the conical vessel, of which C is the vertex, and AB the diameter of its base.



Conceive AI to be horizontal, and produce the axis CD to meet the horizontal line AI in the point I; then is AIC the angle of inclination between the axis and the horizontal line AI.

Let G be the point in the axis through which the plane of the required section is supposed to pass, and through G draw the straight line EF parallel to AB, and GH perpendicular to AI; then is EF the diameter of the section, and GH the perpendicular depth of its centre of gravity below A, the highest particle of the fluid.

Put $R = AD$, the radius of the base of the conical vessel,
$H = CD$, the axis or height,
$r = EG$, the radius of the section on which the pressure is a maximum,
$a =$ the area of the section,
$d = GH$, the perpendicular depth of its centre of gravity,
$p =$ the pressure perpendicular to its surface,
$\phi = AIC$, the angle of inclination between the axis of the cone and the horizon,
$x = CG$, the distance between the section and the vertex of the cone,
and $s =$ the specific gravity of the fluid.

Then, because of the rightangled triangle ADI, and from the principles of Plane Trigonometry, we have

$$R : DI :: \tan.\phi : 1,$$

and from this, we obtain

$$DI = R \cot.\phi,$$

consequently, by adding the axis, we get

$$CI = R \cot.\phi + H,$$

and again by subtraction, it is

$$GI = R \cot.\phi + H - x.$$

But the triangle GHI, is by construction right angled at H; therefore, by Plane Trigonometry, we have

$$GH = d = \{R \cot.\phi + H - x\} \sin.\phi.$$

Again, the triangles CDA and CGE are similar to one another; therefore, by the property of similar triangles, we have

$$H^2 : R^2 : : x^2 : r^2 ;$$

or from this, by equating and dividing, we get

$$r^2 = \frac{R^2 x^2}{H^2} ;$$

consequently, by the principles of mensuration, the area of the section, on which the pressure is proposed to be a maximum, becomes

$$a = \frac{3.1416 R^2 x^2}{H^2} ;$$

therefore, the pressure upon its surface is

$$p = ads = \frac{3.1416 R^2 x^2 s}{H^2} \times \{R \cot.\phi + H - x\} \sin.\phi.$$

Now, because the whole of the quantities which enter this equation are constant, excepting x, and the bracketted expression $\{R \cot.\phi + H - x\}$ which is affected by it; it follows, that the value of p varies as $x^2 \{R \cot.\phi + H - x\}$, and consequently, is a maximum, when the quantity which limits its variation is a maximum; hence we have

$$x^2 \{R \cot.\phi + H - x\} = \text{a maximum}.$$

Let the above expression for the maximum be thrown into fluxions, and we shall obtain

$$2 (R \cot.\phi + H) x \dot{x} - 3x^2 \dot{x} = 0 ;$$

therefore, by transposing and expunging the common quantities, we get

$$3x = 2 (R \cot.\phi + H),$$

and finally, by division, we obtain

$$x = \tfrac{2}{3} (R \cot.\phi + H). \qquad (132).$$

169. The practical rule for reducing this equation, may be expressed in words at length in the following manner.

> RULE. *Multiply the natural cotangent of the angle which the axis of the cone makes with the horizon, by the radius of the vessel's base, and to the product add the altitude or axis of the cone; then, two thirds of the sum will give the distance of the section, on which the pressure is a maximum, from the vertex of the cone.*

170. EXAMPLE. A conical vessel whose altitude is 20 inches, and the radius of its base 8 inches, is filled with fluid and so inclined, that its axis makes with the horizontal line passing through the extremity of the diameter of its base, an angle of 48 degrees; on what section parallel to the base is the pressure a maximum?

Here we have given, $R = 8$ inches, $H = 20$ inches, and $\phi = 48$ degrees, of which the natural cotangent is 0.9004 very nearly; consequently, by the rule, we have

$$x = \tfrac{2}{3}\{8 \times 0.9004 + 20\} = 18.1355 \text{ inches.}$$

If therefore, 18.1355 inches be set off from the vertex, a straight line drawn through that point parallel to the base, will be the diameter of the section on which the pressure is a maximum.

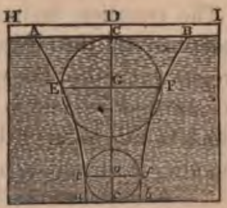
171. In any fluid the particles towards its base support those that are immediately above; these again bear the load above them, and so on to the surface, where the whole mass supports the superincumbent atmosphere. There is therefore a pressure among the successive strata of an homogeneous fluid increasing in exact proportion to the perpendicular depth. Hence a bubble of air or of *steam*, set at liberty far below the surface of water, is small at first, and gradually enlarges as it rises. This phenomenon shows that the compressive power of the fluid slackens by ascent. Experiment and calculation most readily demonstrate the compressibility of water: and the next problem exhibits the striking effects from the increase of pressure at great depths of the sea.

PROBLEM XXVIII.

172. A globular body of condensible and elastic matter, is suffered to ascend vertically from the bottom to the surface of the sea:—

> *It is required to determine its diameter at the surface, the depth of the sea, and the diameter at the bottom being given.*

Let AB be the surface of the sea, ab its bottom, and ecf, ECF, two positions of the globular body in its ascent from the bottom to the surface, and AEea, BFfb the curves described by the extremities of the diameter.



Through G and g, the centre of the globe in the two positions, draw the vertical line Cc, which is manifestly the abscisse to the curve, the radii ge and GE, as well as ca and CA, being ordinates.

Produce the abscisse cC to D, and make CD equal to the height of a column of sea water, which would be in equilibrio with the pressure of the atmosphere; through the point D draw the straight line HI parallel to the surface of the sea, and HI will manifestly be the asymptote to the curve described by the extremities of the diameter.

Put $r = ac$, the radius of the globe at the bottom of the water,
$d = \mathrm{C}c$, the depth of the water at the place of immersion,
$h = \mathrm{CD}$, the height of a column of water equal to the weight of the atmosphere,
$x = \mathrm{CG}$, any abscissa,
$y = \mathrm{GE}$, the corresponding ordinate, or radius of the globe.

Then, because the magnitude of the globular body, is inversely as the density, (the weight and the quantity of matter remaining the same,) and the density is directly as the pressure; it follows, that the magnitude of the body at different points of its ascent, is inversely as the pressure at those points, and the pressure is directly as the depth; therefore, we have

$$\mathrm{DC} : \mathrm{DG} :: \mathrm{GE}^3 : \mathrm{C}a^3;$$

but according to the foregoing notation, we have

$$d + h : h + x :: y^3 : r^3;$$

from which, by equating the products of the extremes and means, we get

$$y^3 (h + x) = r^3 (d + h);$$

hence, by division, we obtain

$$y^3 = \frac{r^3 (d + h)}{(h + x)},$$

and by extracting the cube root, it is

$$y = r\sqrt[3]{\frac{(d + h)}{(h + x)}}. \qquad (133).$$

The equation in its present form, exhibits the nature of the curve described by the diameter of the body during its ascent; or it expresses generally, the value of the ordinate or radius corresponding to any depth; but in order to determine the radius at the surface, which is the primary demand of the problem, we must suppose the quantity x to vanish, in which case, the above equation becomes

$$y = r\sqrt[3]{\frac{d + h}{h}}. \qquad (134).$$

173. The practical rule supplied by, or derived from this equation, may be expressed in words at length in the following manner.

RULE. *To the given depth of the sea, add the height of a column of sea water, which is equal to the weight or pressure of the atmosphere; divide the sum by the height of the atmospheric column, and multiply the radius of the body at the bottom of the sea by the cube root of the quotient, and the product will give the radius at the surface.*

174. EXAMPLE. The radius of a globe of elastic and condensible matter, when placed at the depth of 75 fathoms in sea water, is equal to 4 inches; what will be the radius on ascending to the surface, the atmospheric column being equal to 33 feet?

Here we have given $d=75$ fathoms, or $75\times 6=450$ feet; $h=33$ feet; $r=4$ inches; consequently, by the rule, we have

$$y=4\sqrt[3]{\frac{483}{33}}=9.785 \text{ inches nearly.}$$

175. From this it appears, that if a globe of condensible matter, whose radius is 9.785 inches, be immersed in the sea to the depth of 450 feet, its radius will be decreased to 4 inches; this circumstance may suggest some easy and accurate methods of determining the depth of the ocean, when it is so great as to preclude the application of other methods.

176. In order, however, to adapt our equation to the determination of the depth, we must consider the radii at the surface and at the bottom, together with the height of the atmospheric column, to be accurately known at the time of trial; then, by a very obvious transformation, the depth of descent may be ascertained; for let R be substituted instead of y in the foregoing equation, to denote the radius at the surface, and we shall have

$$R=r\sqrt[3]{\frac{(d+h)}{h}},$$

in which equation, d is the unknown quantity.

Let both sides of the equation be divided by r, the radius of the globe at the bottom of the sea, and we shall obtain

$$\frac{R}{r}=\sqrt[3]{\frac{(d+h),}{h}},$$

and cubing both sides, it becomes

$$\frac{R^3}{r^3}=\frac{(d+h)}{h};$$

multiply by h, and we obtain

$$\frac{R^3 h}{r^3} = d + h.$$

and finally, by transposition, we have

$$d = \frac{h(R^3 - r^3)}{r^3}. \qquad (135).$$

177. The practical rule for reducing the above equation, may be expressed in words at length in the following manner.

> RULE. *Multiply the difference of the cubes of the radii, by the height of the atmospheric column, and divide the product by the cube of the lesser radius for the depth required.*

EXAMPLE. The radius of a globe of condensible matter is 10 inches before immersion, and it is suffered to descend so far as to have its radius diminished to 3 inches; required the depth of descent, the atmospheric column at the time of the experiment being equivalent to 33 feet.

Here we have given $R = 10$ inches, $r = 3$ inches, and $h = 33$ feet; therefore, by proceeding according to the rule, we have

$$R^3 - r^3 = 1000 - 27 = 973;$$

consequently, multiplying by 33 feet, we obtain

$$973 \times 33 = 32109,$$

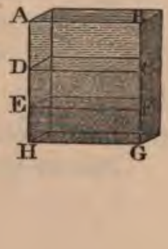
therefore, by division, it is

$$d = \frac{32109}{27} = 1189\tfrac{2}{9} \text{ feet}.$$

PROBLEM XXIX.

178. Let a vessel of any form whatever, whose base is horizontal, be filled with fluids of different densities which do not mix :—

> *It is required to determine the pressure on the bottom of the vessel, supposing the fluids to succeed each other in the order of their densities.*



Let ABGH represent a vertical section of the vessel, containing fluids of different densities or specific gravities, as indicated by the shading of the several strata AC, DF and EG; and for the sake of simplicity of investigation, let the bottom HG be parallel, and the sides AH, BG perpendicular to the horizon. Then are AB, DC and EF, the respective surfaces of the several fluids, as *mercury*, *water*, and *olive oil*, also parallel to the horizon; for, as we have elsewhere stated:—

> *The common surface of two fluids which do not mix, is parallel to the horizon.*

Now, it is manifest, (since the sides AH and BG are perpendicular to the base HG), that the pressure upon the base HG, is equal to the pressures or weights of the several fluids contained in the vessel; therefore

Put $d =$ EH, the perpendicular depth of the lowest stratum EG,
$d' =$ DE, the perpendicular depth of the middle stratum DF,
$d'' =$ AD, the perpendicular depth of the upper stratum AC,
$p =$ the pressure of the stratum EG upon the line HG,
$p' =$ the pressure of the stratum DF upon the line EF,
$p'' =$ the pressure of the stratum AC upon the line DC; and let s, s' and s'' denote the specific gravities of the respective fluids.

Then, since the pressure upon any surface, is equal to the area of that surface, drawn into the perpendicular depth of its centre of gravity; it follows, that the pressure upon HG, occasioned by the fluid in EG, is

$$p = \text{HG} \times ds,$$

and in like manner, the pressure upon EF, is

$$p' = \text{EF} \times d's',$$

and lastly, the pressure upon DC, is

$$p'' = \text{DC} \times d''s''.$$

But the total pressure upon HG, is manifestly equal to the sum of these pressures; therefore, if P denote the entire pressure on the line HG, we have

$$\text{P} = p + p' + p'' = \text{HG} \times ds + \text{EF} \times d's' + \text{DC} \times d''s'';$$

but the lines HG, EF and DC, are equal among themselves, therefore we get

$$\text{P} = \text{HG}\,(ds + d's' + d''s''). \qquad (136).$$

179. In the preceding investigation, we have considered three fluids of different densities to be contained in the vessel; but the same mode of procedure will extend to any number whatever, and what we have done respecting three fluids is sufficient to discover the law of induction for any other number. It is this:—

The perpendicular pressure upon the horizontal base of a vessel containing any number of fluids of different densities, which do not mix in the vessel:—

> *Is equal to the area of the base, multiplied by the sum of the products of the specific gravities drawn into the altitudes of the several fluids.*

But the pressure upon the base, will manifestly be the same, if we suppose the vessel to be filled with a fluid of uniform density, arising from the composition of the densities of the several fluids according to their magnitudes; or if the magnitudes are equal, the uniform density will be a medium between the several given densities.

180. Example. A cylindrical vessel, whose diameter is 6 and altitude 24 inches, is filled with mercury, water and olive oil, in the following proportions, viz. mercury 7, water 8, and olive oil 9 inches; what is the pressure on the bottom of the vessel, the specific gravities being 13598, 1000 and 915 respectively?

Here, by the principles of mensuration, the area of the bottom of the vessel containing the fluids, is

$$36 \times .7854 = 28.2744 \text{ square inches;}$$

consequently, the pressure produced by the mercury, is

$$p = 28.2744 \times 7 \times 13598 = 2691327.0384,$$

and in like manner, the pressure of the water, is

$$p' = 28.2744 \times 8 \times 1000 = 226195.2,$$

and lastly, the pressure produced by the oil, is

$$p'' = 28.2744 \times 9 \times 915 = 232839.684;$$

and the sum of these is manifestly the whole pressure; hence we get

$$P = 2691327.0384 + 226195.2 + 232839.684 = 3150361.9224.$$

If the pressure as here expressed be divided by 1728, the number of solid inches in a cubic foot, we shall have

$$P = \frac{3150361.9224}{1728} = 1823.1261 \text{ ounces.}$$

181. Again, suppose the dimensions of the vessel to remain as above, and let it be filled with the same fluids in equal quantities; that is, 8 inches of mercury, 8 of water, and 8 of olive oil; what then is the pressure upon the bottom?

Here, by proceeding as above, we have for mercury,

$$p = 28.2744 \times 8 \times 13598 = 3075802.3296;$$

for water, it is

$$p' = 28.2744 \times 8 \times 1000 = 226195.2,$$

and for olive oil, it is

$$p'' = 28.2744 \times 8 \times 915 = 206968.608;$$

hence by summation, the entire pressure on the bottom, is

$$P = 3075802.3296 + 226195.2 + 206968.608 = 3508966.1376,$$

and lastly, dividing by 1728, we obtain

$$P = \frac{3508966.1376}{1728} = 2030.6517 \text{ ounces.}$$

The pressure which we have found in this last instance, is the very same as that which would arise, if the vessel were filled with fluid of a medium density; for we have

$$\tfrac{1}{3}(13598 + 1000 + 915) = 5171 \text{ medium density};$$

hence, the entire pressure on the bottom, is

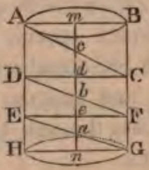
$$P = 28.2744 \times 24 \times 5171 = 3508966.1376;$$

which by division, gives

$$P = \frac{3508966.1376}{1728} = 2030.6517 \text{ ounces, the same as before.}$$

182. Let the conditions of the problem remain as above, and let it be required to determine the pressure on the concave surface of the vessel, and to compare it with that upon the bottom.

Let ABGH, as in the preceding case, represent an upright section of the vessel, of which the base HG is parallel, and the sides AH, BG perpendicular to the horizon; and suppose the fluids of different densities to be contained in the strata AC, DF and EG.



Bisect the surface and base AB and HG, in the points m and n, and join mn; then do the centres of gravity of the several cylindric surfaces occur in that line. Draw the diagonals AC, DF and EG, cutting the vertical line mn in the points c, b and a, which mark the places of the respective centres of gravity.

Put D $=$ AB or HG, the diameter of the vessel containing the fluids,
$\tfrac{1}{2}d = ea$, the depth of the centre of gravity of the lower cylindric surface,
$\tfrac{1}{2}d' = db$, the depth of the middle cylindric surface,
$\tfrac{1}{2}d'' = mc$, the depth of the upper cylindric surface; each of these being referred to the surface of the respective fluid.

Then, the pressures and the specific gravities being denoted as before, by the letters p, p', p'', and s, s', s'', we shall obtain as follows.

According to the principles of mensuration, the circumference of the vessel is expressed by 3.1416 D; consequently the several surfaces, estimated in order upwards, are

$$3.1416\,\text{D}d;\ 3.1416\,\text{D}d', \text{ and } 3.1416\,\text{D}d'';$$

and the corresponding pressures, are

$$p = 1.5708\,\text{D}d^2s;\ p' = 1.5708\,\text{D}d'^2s', \text{ and } p'' = 1.5708\,\text{D}\,d''^2s'';$$

and the total pressure is

$$\text{P}' = (p + p' + p'') = 1.5708\,\text{D}\,(d^2s + d'^2s' + d''^2s''). \quad (137).$$

But the area of the base is expressed by .7854 D²; consequently, the equation numbered (136) becomes

$$\text{P} = .7854\,\text{D}^2\,(ds + d's' + d''s''),$$

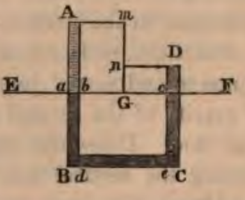
and the pressures on the base, and on the upright surface of the vessel, are to one another as $\text{D}\,(ds + d's' + d''s'') : 2\,(d^2s + d'^2s' + d''^2s''$.

PROPOSITION III.

183. If two fluids of different densities or specific gravities, communicate with one another through a bent tube or otherwise, and remain in a state of equilibrium:—

> *The perpendicular altitudes of these fluids above their common surface, will vary inversely as their specific gravities.*

Let ABCD be a tube, through the bent arms of which, two fluids of different specific gravities, communicate with one another in the common surface ab, and suppose the horizontal plane EF to pass through the surface of communication.



Take G the centre of gravity of the plane EF, and through G draw the vertical line Gm, meeting Am and Dn respectively in the points m and n; then, because the lines Am and Dn are parallel to the horizon, mG and nG are the perpendicular depths of the plane EF, below the surfaces of the fluids at A and D.

Now it is manifest, that since the part of the plane ab, which is contiguous to the common surface of the fluids, is sustained in its place by the downward pressure of the lighter fluid in Ab, and by the upward pressure of the heavier fluid in DCB; it follows, that these pressures are equal to one another. But because the plane EF is parallel to the horizon, the pressure of the fluid in bB, is equal and opposite to the pressure in cC, the fluid in de serving no other purpose than for mutually transmitting the opposing pressures; consequently, the pressure of the lighter fluid in Ab, is counterpoised by the pressure of the heavier fluid in Dc, the fluid in $c$CB$b$ serving only as a medium of communication.

Put $d = m$G, the perpendicular depth of the plane EF, below the surface of the lighter fluid at A,

$\delta = n$G, the perpendicular depth of the plane EF, below the surface of the heavier fluid at D,

$p =$ the pressure on the plane, occasioned by the lighter fluid in Ab,

$p' =$ the pressure on the plane, occasioned by the heavier fluid in Dc; and let s and s' represent the corresponding specific gravities of the lighter and the heavier fluids.

Then, since the pressures on the plane, occasioned by the actions of the two fluids, are respectively as the depths of the centre of gravity, and the specific gravity of the fluids jointly; it follows, that

$$p : p' :: ds : \delta s';$$

but according to the conditions of the proposition, these pressures are equal to one another; hence we have

$$ds = \delta s'; \qquad (138).$$

and by converting this equation into an analogy, it becomes

$$d : \delta :: s' : s;$$

hence, the truth of the proposition is rendered manifest.

Let both sides of the equation numbered (138), be divided by s, the specific gravity of the lighter fluid, and we shall obtain

$$d = \frac{\delta s'}{s}. \qquad (139).$$

184. Hence, in order to determine the perpendicular depth, or altitude of a column of the lighter fluid, that will balance or keep in equilibrio a given column of the heavier; we must observe the following practical rule.

RULE. *Multiply the altitude of the heavier fluid by its specific gravity, and divide the product by the specific gravity of the lighter fluid, for the altitude sought.*

185. **Example.** The height of a column of mercury is 30 inches, or $2\frac{1}{2}$ feet, and its specific gravity 13598 ounces per cubic foot; what is the height of the equilibrating column of water, its specific gravity being 1000 ounces per cubic foot?

The operation performed according to the rule, is as below.

$$d = \frac{13598 \times 2\frac{1}{2}}{1000} = 33.995 \text{ feet.}$$

186. A column of mercury 30 inches or $2\frac{1}{2}$ feet in perpendicular height, is found to equiponderate with the atmosphere in a medium state of temperature; consequently, a column of water 33.995 or 34 feet nearly in perpendicular height, will produce the same effect; it is therefore manifest, that water will ascend in a vacuum tube, to the height of about 34 feet, by means of the pressure of the atmosphere; and on this principle depends the operation of the sucking pump, to which we shall have occasion to advert in another place.

187. If in the equation numbered (139), the specific gravities are equal to one another, that is, if $s = s'$, then $d = \delta$, or the perpendicular altitudes of two fluids whose specific gravities are equal, are also equal, the fluids being supposed to communicate with one another in the arms of a bent tube, whatever may be the shape or position of the arms through which the communication takes place.

188. This explains the reason why the surfaces of small pools or collections of water near rivers, are always on a level with the surfaces of the rivers, when there is any subterraneous communication between them.

189. It is on this principle also, that water may be conveyed from any one place, to any other place of the same or a less elevation; for by means of pipes, a communication can be opened between the places, and whatever may be the number of elevations and depressions, or deviations from the same vertical plane, and whatever may be the distance from the source to the point of discharge, the water will continue to flow along the communicating vessels, provided always, that none of the intervening elevations exceeds the level of the stagnant fluid, or the source from which the water flows.

190. When the point to which the water is conveyed, is of the same altitude as that from which it proceeds, the surface will be in a state of quiescence; but if the point of discharge be lower than the point of supply, the fluid, by endeavouring to rise to the same level, will cause a stream to flow.

It is by this property of fluids endeavouring to rise to the same level, that large towns and cities are supplied with water from a distance; the city of Edinburgh, in Scotland, is supplied in this way; but the successful execution of all such complicated and elaborate undertakings, requires an immense outlay of capital, directed by the skill and judgment of the most eminent engineers.*

191. The principle which we have demonstrated in the foregoing proposition, with respect to two fluids of different specific gravities, may in like manner, be shown to obtain with any number of fluids whatever; a separate demonstration, however, would here be out of place, we shall therefore content ourselves with the general enunciation; but the reader may, for his own satisfaction and improvement, supply the demonstration; the enunciation is as follows.

If any number of fluids of different specific gravities, communicate with one another through the arms of a bent tube, and remain in equilibrio :—

> *The sums of the products of their perpendicular heights and specific gravities, in each branch of the communicating tube, shall be equal to one another.*

Various interesting and important problems might be proposed, on the principle of fluids of different densities, communicating with one another in the opposite branches of a bent tube; but our limits will only admit of the following, which on account of its elegance, is worthy of a place in our present inquiry.

PROBLEM XXX.

192. The ratio of the specific gravities of two fluids being given, if equal quantities of the fluids be poured into a circular tube of uniform diameter :—

> *It is required to determine their position when in a state of equilibrium.*

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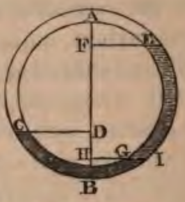
* Baths and aqueducts contributed largely, in the Roman empire, to the wealth and comfort of the meanest citizen, whether in the solitudes of Asia and Africa, or the well watered provinces of the west. The splendour, the wealth, even the existence of those numerous and populous cities which are now no more, was derived from such artificial supplies of a perennial stream of fresh water. The boldness of the enterprise and the solidity of the workmanship may be judged of by those of Spoleto, Metz, Segovia, &c. The aqueduct of Troas, constructed partly at the expense of the generous Atticus, is but a solitary instance of the spirit of those PATRICIANS who were not afraid of displaying to the world that they had the wisdom to conceive and wealth to accomplish the noblest undertakings.—See *Gibbon's Decline and Fall*, vol. i. chap. II.



Let AB be the interior diameter of the circular tube, and CI, IE the spaces occupied by the fluids when they have attained the state of quiescence, GI being the common surface, or the plane in which the communication occurs.

Through the common surface IG, the surface of the heavier fluid at C, and the surface of the lighter at E, draw the lines GH, CD and EF respectively parallel to the horizon, and meeting the diameter AB at right angles in the points H, D and F; then is DH the vertical altitude of the heavier fluid, above the common surface IG, and FH is the vertical altitude of the lighter fluid, as referred to the same plane.

Put $d =$ FH, the perpendicular altitude of the lighter fluid,
$\delta =$ DH, the perpendicular altitude of the heavier fluid,
$s =$ the specific gravity of the lighter fluid,
$s' =$ the specific gravity of the heavier fluid,
$\phi =$ CG or GE, the portion of the circular tube which is occupied by each of the fluids,
$x =$ CB, the number of degrees between the highest and lowest points of the heavier fluid.

Then, because by the preceding proposition, the perpendicular altitudes of two fluids of different densities, which communicate with one another through the branches of a bent tube, are inversely as the densities or specific gravities; it follows, that

$$d : \delta :: s' : s,$$

and from this analogy, by making the product of the mean terms equal to the product of the extremes, we obtain

$$ds = \delta s';$$

the very same result as equation (138), and if both sides be divided by s, we shall obtain

$$d = \frac{\delta s'}{s}.$$

Now, if the specific gravity of the lighter fluid be expressed by unity, while that of the heavier is denoted by m; then the above equation becomes

$$d = m\delta. \qquad (140).$$

By referring to the diagram, it will readily appear, that the space occupied by both the fluids, when in a state of equilibrium, is represented by

$$\text{CGE} = 2\phi,$$

and according to our notation, it is shown that

$$\text{CB} = x;$$

consequently, by subtraction, we obtain

$$\text{BG} = \phi - x, \text{ and BGE} = 2\phi - x.$$

But by the nature of the circle and the principles of Plane Trigonometry, it is manifest, that BD is the versed sine of the arc BC; BH the versed sine of the arc BG, and BF the versed sine of the arc BGE; therefore, by re-establishing the respective symbols, we shall obtain

$$\text{FH} = d = \text{vers.}(2\phi - x) - \text{vers.}(\phi - x),$$

and by a similar subtraction, we get

$$\text{DH} = \delta = \text{vers.}x - \text{vers.}(\phi - x).$$

Let both sides of this equation be multiplied by m, the co-efficient of δ in equation (140), and we shall have

$$m\delta = m\,\{\text{vers.}x - \text{vers.}(\phi - x)\};$$

consequently, by comparison, we obtain

$$\text{vers.}(2\phi - x) - \text{vers.}(\phi - x) = m\{\text{vers.}x - \text{vers.}(\phi - x)\}.$$

Now, in order to simplify the reduction of this equation, it will be proper to substitute for the several versed sines of which it is composed, their corresponding values in terms of the radius and cosines; for, by such a substitution we obtain

$$\cos.(\phi - x) - \cos.(2\phi - x) = m\,\{\cos.(\phi - x) - \cos.x\};$$

but by the arithmetic of sines, we have

$$\cos.(\phi - x) = \cos.\phi\cos.x + \sin.\phi\sin.x, \text{ and } \cos.(2\phi - x) = \cos.2\phi\cos.x + \sin.2\phi\sin.\phi;$$

consequently, by substitution, we get

$$\cos.\phi\cos.x + \sin.\phi\sin.x - \cos.2\phi\cos.x - \sin.2\phi\sin.x = m\,\{\cos.\phi\cos.x + \sin.\phi\sin.x - \cos.x\};$$

and from this, by transposition, we have

$$m\cos.x - \cos.2\phi\cos.x + \sin.2\phi\sin.x + (m-1)(\cos.\phi\cos.x + \sin.\phi\sin.x).$$

Let all the terms of this equation be divided by $\cos.x$, and we shall obtain

$$m = \cos.2\phi + \sin.2\phi\tan.x + (m-1)(\cos.\phi + \sin.\phi\tan.x);\ *$$

therefore, by separating and transposing the terms, we get

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* It is demonstrated by the writers on Analytical Trigonometry, that the sine divided by the cosine to the same radius, is equal to the tangent; hence we have

$$\tan.x = \frac{\sin.x}{\cos.x}.$$

$$-\sin.2\phi\tan.x-(m-1)\sin.\phi\tan.x=\cos.2\phi+(m-1)\cos.\phi-m,$$

and finally, by division, we obtain

$$\tan.x=-\frac{\cos.2\phi+(m-1)\cos.\phi-m}{\sin.2\phi+(m-1)\sin.\phi}. \qquad (141).$$

193. We believe that Mr. Barclay's *Hydrostatic Quadrant*, for finding the altitude of the heavenly bodies when the horizon is obscure, is founded on principles similar to those propounded in this problem, and expressed in the above equation; but it would be improper in this place to attempt a delineation of this instrument; it will therefore suffice, to illustrate the reduction of the above formula, by a numerical example performed according to the directions contained in the following practical rule.

RULE. *From the specific gravity of the heavier fluid, subtract unity; multiply the remainder by the natural cosine of the circular space occupied by each fluid; to the product add the natural cosine of the circular space occupied by both fluids; then, from the sum subtract the greater specific gravity, and the remainder will be the dividend.*

Again. *From the specific gravity of the heavier fluid, subtract unity; multiply the remainder by the natural sine of the circular space occupied by each fluid; then, to the product add the natural sine of the circular space occupied by both fluids, and the sum will be the divisor.*

Lastly. *Divide the dividend by the divisor, and the quotient will give the natural tangent of a circular arc, which being found in the tables, enables us to assign the position of the fluids when in a state of equilibrium.*

194. EXAMPLE. Suppose that 100 degrees of the inner circumference of a circular tube, exhibits equal quantities of mercury and water, whose specific gravities are to one another, very nearly, as 14 to 1; it is required to assign the position of the fluids, with respect to the vertical diameter of the tube, when they are in a state of equilibrium with each other; that is, when they excite equal pressures on the plane passing through their common surface?

Here we have given $m=14:\phi=50°$, its natural sine and cosine .76604 and .64279 respectively; $2\phi=100°$, its natural sine .98481, and its cosine $-.17365$; therefore, by the rule, we have

For the dividend,

$$-\cos.2\phi+(m-1)\cos.\phi-m=-.17365+13\times.64279-14=-5.81738,$$

For the divisor,

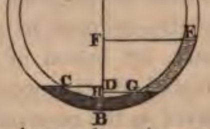
$$-\sin.2\phi-(m-1)\sin.2\phi=-.98481-13\times.76604=-10.94333;$$

consequently, by division, we obtain

$$\tan.x=\frac{-5.81738}{-10.94333}=.53159=\text{nat. tan. } 27^\circ\, 59'\, 41''.$$

195. Having discovered the value of x by the preceding operation, the actual position of the fluids, with respect to the vertical diameter of the tube, may from thence be very easily exhibited.

Let ACBE represent a circular tube of glass, or some other transparent matter, partly filled with mercury and water, in such quantities, that when the tube is retained in a vertical plane, and the fluids in equilibrio, a space equivalent to fifty degrees of the inner surface comes in contact with each; it is therefore required to assign the actual position of the fluids.



Draw the vertical diameter AB, and from the point B where it meets the inner circumference of the tube, set off BC from a scale of chords, equal to 27° 59′ 41″; then, take 50° in the compasses, and setting one foot on C extend the other to G, thereby marking off the space occupied by the mercury, including the lowest portion of the tube; then, with the same extent of the compasses, set off GE the space occupied by the water, and the position of the fluids is from thence determined.

Through the points C, G and E draw the straight lines CD, GH and EF respectively parallel to the horizon, and meeting AB the vertical diameter perpendicularly in the points D, H and F; then are DH and FH the perpendicular altitudes of the mercury and the water, as referred to the plane passing through their common surface at G; and BD, BF are the respective altitudes, as referred to the vertical diameter AB.

Now, according to the question, the specific gravity of the mercury, is fourteen times greater than that of the water; and by the third proposition preceding, the perpendicular altitudes are inversely as the specific gravities; consequently, FH must be fourteen times greater than DH; when the positions of the fluids are properly determined; let us therefore inquire if this be the case.

It has been found above, that BC is equal to 27° 59′ 41″, and by construction CG and GE are each equal to 50 degrees; consequently, $BG=50^\circ-27^\circ\, 59'\, 41''=22^\circ\, 0'\, 19''$, and $BE=50^\circ+22^\circ\, 0'\, 19''=72^\circ\, 0'\, 19''$; hence we have

BE $= 72^\circ\, 0'\, 19''$ - - - nat. vers. $= .69106$,
BC $= 27^\circ 59'\, 41''$ - - - nat. vers. $= .11700$,
BG $= 22^\circ\, 0'\, 19''$ - - - nat. vers. $= .07285$;

consequently, by subtraction, we obtain

FH $=$ BF $-$ BH; that is, FH $= .69106 - .07285 = .61821$,

and after the same manner, we get

DH $=$ BD $-$ BH; that is, DH $= .11700 - .07285 = .04415$;

therefore, according to the proposition, we have

$.04415 : .61821 :: 1 : 14$ very nearly.

196. The above result has been obtained on the supposition, that the fluids enclosed in the tube are mercury and water; mercury, on account of its great density and high degree of purity, is very frequently enclosed in tubes and applied to permanent purposes; but water, by reason of its liability to become putrid, is not so well adapted for the occasion, and consequently, is seldom or never employed in the construction of philosophical instruments.

197. There are however, several other fluids that do not partake of the putrescent nature of water, and whose specific gravities may be either greater or less according to the required circumstances; some of these, on account of the colouring matter which they contain, are very convenient, and from the length of time that they retain their spirit and purity, are generally employed in preference to others, which do not possess these very requisite and important characteristics.

198. Now, the result of our investigation, as we have already observed, is only applicable in the case of mercury and water; at least, equation (141) implies, that the specific gravity of one of the fluids is expressed by unity, and this, according to our present standard, can only obtain when water becomes the subject of reference.

199. It is therefore necessary, in order that our formula may apply to fluids without distinction, to bring it into a general form, and this is very easily done; for we have shown in the preceding investigation, that

$$d = \frac{\delta s'}{s},$$

but it has also been shown, that

$$d = \text{vers.}\,(2\phi - x) - \text{vers.}\,(\phi - x);$$

consequently, by comparison, we obtain

$$\frac{\delta s'}{s} = \text{vers.}(2\phi - x) - \text{vers.}(\phi - x);$$

and moreover, it has been further proved, that

$$\delta = \text{vers.}x - \text{vers.}(\phi - x);$$

therefore, multiplying both sides by $\frac{s'}{s}$, we shall get

$$\frac{\delta s'}{s} = \frac{s'}{s}\{\text{vers.}x - \text{vers.}(\phi - x)\};$$

let these two values of $\frac{\delta s'}{s}$ be compared with one another, and we shall obtain

$$\text{vers.}(2\phi - x) - \text{vers.}(\phi - x) = \frac{s'}{s}\{\text{vers.}x - \text{vers.}(\phi - x)\};$$

or multiplying by s, we have

$$s\,\text{vers.}(2\phi - x) - s\,\text{vers.}(\phi - x) = s'\,\text{vers.}x - s'\,\text{vers.}(\phi - x).$$

Now, by substituting for the several versed sines, their values in terms of the cosines and radius, we shall obtain

$$s\{\cos.(\phi - x) - \cos.(2\phi - x)\} = s'\{\cos.(\phi - x) - \cos.x\}$$

from which, according to the arithmetic of sines, we get

$$s\{\cos.\phi\cos.x + \sin.\phi\sin.x - \cos.2\phi\cos.x - \sin.2\phi\sin.x\} = s'\{\cos.\phi \cos.x + \sin.\phi\sin.x - \cos.x\}.$$

Let all the terms of this equation be divided by $\cos.x$, and it becomes transformed into

$$s\cos.\phi + s\sin.\phi\tan.x - s\cos.2\phi - s\sin.2\phi\tan.x = s'\cos.\phi + s' \times \sin.\phi\tan.x - s';$$

therefore, by bringing to one side, all the terms that involve $\tan.x$, we shall have

$$-\sin.\phi\,(s' - s)\tan.x - s\sin.2\phi\tan.x = s\cos.2\phi + (s' - s)\cos.\phi - s';$$

hence, by division, we shall obtain

$$\tan.x = -\frac{s\cos.2\phi + (s' - s)\cos.\phi - s'}{s\sin.2\phi + (s' - s)\sin.\phi}. \qquad (142).$$

If the equation which we have just obtained, be compared with that numbered (141), it will readily appear, that the one might have been deduced immediately from the other, by simply substituting s' for m, and s for unity, in the several terms of the numerator and denominator; but in order to render the formation of the formula more intelligible, we have thought proper to trace the steps throughout.

200. The practical rule for reducing the above equation, will require a different mode of expression from that which we have given in the rule to equation (141), but it will not be more operose; the rule is as follows.

RULE. *Multiply the difference of the given specific gravities, by the natural cosine of the circular space in contact with one of the fluids; to the product, add the natural cosine of the whole circular space drawn into the less specific gravity, and from the sum subtract the greater specific gravity for a dividend.*

Again. *Multiply the difference between the specific gravities, by the natural sine of the circular space in contact with one of the fluids, and to the product, add the natural sine of the whole circular space drawn into the less specific gravity, and the sum will be the divisor.*

Lastly. *Divide the dividend by the divisor, and the quotient will give the natural tangent of a circular arc, which being found in the tables, enables us to assign the actual position of the fluids when in a state of equilibrium.*

201. EXAMPLE. On the inner surface of a circular tube containing mercury and rectified alcohol, it is observed, that when the tube is held in a vertical plane, and the fluids in a state of equilibrium, a space of 75 degrees of the circumference, is occupied by, or in contact with each fluid; it is required to determine the position of the fluids at the instant of observation, their specific gravities being 14000 and 829 respectively?

In this example there are given $s'=14000$; $s=829$; $\phi=75^\circ$, its natural sine and cosine equal to .96593 and .25882; $2\phi=150^\circ$, its natural sine being .50000, and its cosine $-.86603$; consequently, by proceeding according to the directions contained in the foregoing rule, we shall obtain

For the dividend $-s\cos.2\phi+(s'-s)\cos.\phi-s'=-829\times.86603+(14000-829)\times.25882-14000=-11309.02065$;

For the divisor $-s\sin.2\phi-(s'-s)\sin.\phi=-829\times.50000-(14000-829)\times.96593=-13136.76403$;

consequently, by division, we obtain

$$\tan.x=\frac{-11309.02065}{-13136.76403}=.86086=\text{nat. tan. }40^\circ\,43'\,25''.$$

202. The positions of the fluids in this example, are manifestly very different from what they are in the preceding, the point F in the vertical diameter falling on the other side of the centre; but in this case, we shall leave the construction for the reader's amusement, and proceed to inquire what changes the general formula will undergo, in

consequence of certain assumed spaces of the inner surface, being in contact with each of the contained fluids.

If $\phi =$ half a right angle, that is, if each fluid cover a space of 45 degrees; then $2\phi = 90°$, and consequently, $\sin.2\phi = 1$ and $\cos.2\phi = 0$, while $\sin.\phi = \frac{1}{2}\sqrt{2}$, and $\cos.\phi = \frac{1}{2}\sqrt{2}$; therefore, by substitution, equation (142) becomes

$$\tan.x = -\frac{\frac{1}{2}(s' - s)\sqrt{2} - s'}{\frac{1}{2}(s' - s)\sqrt{2} + s}. \qquad (143).$$

Now, let the fluids be mercury and rectified alcohol, as in the preceding example, then we shall have

$$\tan.x = -\frac{\frac{1}{2}(14000 - 829)\sqrt{2} - 14000}{\frac{1}{2}(14000 - 829)\sqrt{2} + 829} = .55238,$$

which answers to the natural tangent of 28° 55′ nearly.

Again, if $\phi =$ a right angle, that is, if each fluid cover a space of 90 degrees on the inner surface of the tube; then, $2\phi = 180°$, of which the sine and cosine are respectively 0 and -1, while the sine and cosine of ϕ, are respectively 1 and 0; consequently, by substitution, equation (142) becomes

$$\tan.x = \frac{s' + s}{s' - s}. \qquad (144).$$

203. This is a very neat and obvious expression, and the practical rule derived from it, may be enunciated in the following manner.

> Rule. *Divide the sum of the specific gravities of the two fluids by their difference, and the quotient will give the natural tangent of an arc, which being estimated from the lowest point of the tube, will indicate the highest point of the heavier fluid.*

If therefore, the contained fluids be mercury and rectified alcohol, as in the preceding cases, we shall have

$$\tan.x = \frac{14000 + 829}{14000 - 829} = 1.12587,$$

which answers to the natural tangent of 48° 23′ 19″.

We might assume other particular values of the spaces in contact with the fluids, and thereby deduce corresponding forms of the equation; but what we have already done on this subject is quite sufficient.

CHAPTER VIII.

OF THE PRESSURE OF NON-ELASTIC FLUIDS UPON DYKES, EMBANKMENTS, OR OTHER OBSTACLES WHICH CONFINE THEM, WHETHER THE OPPOSING MASS BE SLOPING, PERPENDICULAR OR CURVED, AND THE STRUCTURE ITSELF BE MASONRY OR OF LOOSE MATERIALS, HAVING THE SIDES ONLY FACED WITH STONE.

1. OF FLUID PRESSURE AGAINST MASONIC STRUCTURES.

204. Before we proceed to develope the theory of *Floatation*, and to explain the method of weighing solid bodies by immersing them in, or otherwise comparing them with liquids; it is presumed that it will not be considered out of place, to take a brief survey of the circumstances attending the pressure of non-elastic fluids, when exerted against dykes or other obstacles, that may be opposed to the efforts which they make to spread themselves.

This is an interesting and important subject in the doctrine of *Hydraulic Architecture*, and since the principles upon which it is founded, depend in a great measure on *Hydrostatic* pressure, it cannot properly be omitted in unfolding the elementary departments of the Mechanics of Fluids, which come so directly before our view in what is called *level cutting* in the practice of canal making. Every one knows that in cutting a canal, no further excavation is required than that which will hold the water at a given depth and breadth; when a bank is made on both sides with the earth excavated, the level surface of the canal may be elevated above the natural surface of the adjacent land, and in this case great part of the cost of excavation will be saved. But when the canal is to be carried along wholly within embankments, too much attention cannot be paid to the principles of fluid pressure, if we would avoid unnecessary expense, and at the same time complete the work with systematic regard to its permanent durability; this therefore is the object of the present section, intended as a preliminary article to our Inland Navigation, which will consequently form a part of Hydraulic Architecture.

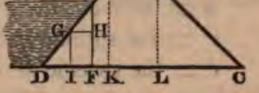
205. When an incompressible and non-elastic fluid presses against a dyke, mound of earth, or any other obstacle that it endeavours to displace, there are two ways in which the obstacle thus opposed may yield to the effort of the fluid.

1. *It may yield by turning upon the remote extremity of its base.*
2. *It may yield by sliding along the horizontal plane on which it stands.*

In either case, the effort to overcome the obstacle, arises from the force which the fluid exerts in a horizontal direction; and the stability of the obstacle, or the resistance which it opposes to being overcome or displaced, arises from its own weight, combined with the vertical pressure of the fluid upon its sloping surface.

206. When the vertical pressure of the fluid is considered, the investigation, as well as the resulting formulæ, are necessarily tedious and prolix; but when the effect of the vertical pressure is omitted, the subject becomes more easy, and the computed dimensions are better adapted for an effectual resistance; but in order to render the investigation general, it becomes necessary to include its effects.

Now, it is manifest from the nature of the inquiry, that when an equilibrium obtains between the opposing forces, the momentum of the horizontal pressure must be equal to the momentum of the vertical pressure, together with the weight of the body on which the pressure is exerted; and for the purpose of showing when this condition takes place, let ABCD represent a vertical section of the dyke, whose resistance is opposed to the pressure of the stagnant fluid, of which the surface is ME and the perpendicular depth EF.



Let AB and DC be parallel to the horizon, and consequently parallel to one another; and from the points E, A and B, demit the straight lines EF, AK, and BL, respectively perpendicular to DC the base of the section.

Take EG any small portion of the sloping side AD, and through the point G, draw the lines GH and GI, respectively parallel and perpendicular to the horizon, constituting the similar triangles EHG, EFD and GID.

The figure being thus prepared, it only remains to establish the proper symbols of reference, before proceeding with the investigation.

Put $b =$ DC, the breadth of the section's base, or the thickness of the dyke at the foundation,
$D =$ AK or BL, the perpendicular altitude or height of the section,
$d =$ EF, the perpendicular depth of the fluid whose surface is at EM,
$\delta =$ DF, the distance between the near extremity of the base at D, and the perpendicular EF,
$c =$ DK, the measure of the slope AD, or the distance between the near extremity of the base at D, and the perpendicular from the extremity of the opposite side at A,
$e =$ CL, the distance between the remote extremity of the base at C, and the perpendicular from the extremity of the opposite side at B, or the measure of the slope BC,
$a =$ ABCD, the area of a vertical section of the obstacle to be displaced,
$p =$ the horizontal pressure of the fluid on the increment of EG,
$f =$ the force with which the horizontal pressure operates to overcome the resistance of the dyke,
$m =$ the momentum of that force,
$p' =$ the vertical pressure of the fluid on the increment of EG,
$f' =$ the force with which the vertical pressure resists the displacement of the obstacle,
$m' =$ the momentum of that force,
$w =$ the symbol which denotes the weight of the dyke or obstacle of resistance,
$F =$ the force with which it opposes the horizontal pressure of the fluid,
$M =$ the momentum of that force,
$s =$ the specific gravity of the fluid,
$s' =$ the specific gravity of the dyke, or opposing body,
$z =$ EG, any small portion of the sloping side AD on which the fluid presses,
$\dot{z} =$ the increment or fluxion of that portion,
$y =$ EH, the perpendicular depth of the point G,
$\dot{y} =$ the increment or fluxion of y,
$x =$ GH, the ordinate or horizontal distance,
and $\dot{x} =$ the increment or fluxion of the horizontal ordinate or distance GH.

Then, since the pressure upon any line or surface, is equal to, or expressed by the magnitude of that line or surface, multiplied by the perpendicular depth of its centre of gravity, and again by the specific gravity of the fluid; it follows, that the horizontal pressure on the increment of EG, is

$$p = sy\dot{z};$$

but by the principles of mechanics, the aggregate or accumulated force, with which the horizontal pressure operates to overturn or remove the dyke, is

$$f = s\int y\dot{z} \times \frac{\dot{y}}{\dot{z}} = s\int y\dot{y},$$

and by taking the fluent of this, it is

$$f = \tfrac{1}{2}sy^2.$$

But the perpendicular distance from E, at which this force must be applied, is manifestly equal to $\frac{2}{3}y$; for the centre of gravity of the triangle EHG, occurs in the horizontal line passing through that point; therefore, the length of the lever on which the force operates to overturn the dyke is

$$y - \tfrac{2}{3}y = \tfrac{1}{3}y;$$

consequently, for the momentum of the force, we have

$$m = \tfrac{1}{2}sy^2 \times \tfrac{1}{3}y = \tfrac{1}{6}sy^3,$$

and when y becomes equal to d, the whole height of the fluid, it is

$$m = \tfrac{1}{6}sd^3. \qquad (145).$$

Again, the vertical pressure exerted by the fluid on the increment of EG, is obviously equal to the weight of the incumbent column; that is

$$p' = sy\dot{x},$$

and this pressure expresses the force, with which the fluid operates vertically to retain the obstacle in its position, or to prevent it from rising to turn about the point C; consequently,

$$p' = f' = sy\dot{x}.$$

Now, the length of the lever on which this force acts, is evidently equal to IC, the distance between the fulcrum C, and the point I, where the perpendicular passing through G cuts the base DC; but IC according to the figure, is equal to DC — DF + IF; that is

$$\text{IC} = b - \delta + x;$$

consequently, the momentum of the force f', is

$$m' = sy\dot{x}(b - \delta + x),$$

or taken collectively, the momentum on EG, is

$$m' = s\int y\dot{x}(b - \delta + x);$$

but by reason of the similar triangles EHG and EFD, we have the following proportion, viz.

$$d : \delta :: y : x,$$

from which we obtain

$$x = \frac{\delta y}{d},$$

and because the fluxions of equal quantities are equal, it is

$$\dot{x} = \frac{\delta \dot{y}}{d}.$$

Let these values of x and $\dot{x}$, be substituted instead of them in the preceding value of m', and we shall obtain

$$m' = s\int \left\{ b - d + \frac{\delta y}{d} \right\} \times \frac{\delta y \dot{y}}{d};$$

consequently, by taking the fluent, it becomes

$$m' = \frac{s \delta y^2}{d} \left\{ \tfrac{1}{2}b - \tfrac{1}{2}\delta + \frac{\delta y}{3d} \right\},$$

there being no correction, since the whole expression becomes equal to nothing when y is equal to nothing.

When y becomes equal to d the whole perpendicular height of the fluid, then the foregoing value of m' becomes

$$m' = s \delta d (\tfrac{1}{2}b - \tfrac{1}{6}\delta). \qquad (146).$$

The foregoing equations (145) and (146), exhibit the horizontal and vertical momenta of the pressure exerted by the fluid on the sloping side of the obstacle; and it is manifest from the nature of their action, that they operate in opposition to one another; the horizontal pressure, endeavouring to turn the body round the point C as a fulcrum or centre of motion, and the vertical pressure tending to turn it the contrary way round the same point, or otherwise to render it more stable and firm on its foundation.

208. But the stability of the dyke is farther augmented by means of its own weight, which being conceived to be collected into its centre of gravity, opposes the horizontal pressure of the fluid with a force, which is equivalent to its own weight drawn into a lever, whose length is equal to the perpendicular distance between the centre of motion, and a vertical line passing through the centre of gravity of the section ABCD.

Now, it is manifest from the principles of mensuration, that if the transverse section of the dyke be uniform throughout, the weight is proportional to the area of the section, multiplied into the specific gravity of the material of which it is composed, and again into its length; but the length of the dyke is the same as the length of the fluid which it supports; consequently, the weight is very properly represented by the area of the section and the specific gravity of the material; thus we have

$$w = as'. \qquad (147).$$

But according to the writers on mensuration, the area of the trapezoid ABCD, is equal to the sum of the parallel sides AB and DC, drawn into half the perpendicular distance AK or BL; hence we have

$$a = (\mathrm{AB} + \mathrm{DC}) \times \tfrac{1}{2}\mathrm{AK},$$

but by the foregoing notation, it is

$$\mathrm{AB} = b - (c + e);$$

consequently, by addition, we have

$$\mathrm{AB} + \mathrm{DC} = 2b - (c + e);$$

therefore, the area of the section is

$$a = \tfrac{1}{2}\mathrm{D}(2b - c - e);$$

let this value of a be substituted instead of it in the equation marked (147), and it becomes

$$w = \tfrac{1}{2}\mathrm{D}s'(2b - c - e);$$

but the weight of the dyke is equivalent to the force whose momentum, combined with that of the vertical pressure of the fluid, counterpoises the momentum of the horizontal pressure, which force we have represented by F; hence we have

$$\mathrm{F} = \tfrac{1}{2}\mathrm{D}s'(2b - c - e),$$

and the momentum of this force, is

$$l\mathrm{F} = \mathrm{M} = \tfrac{1}{2}\mathrm{D}ls'(2b - c - e),$$

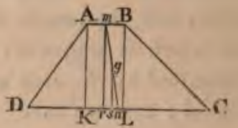
where l denotes the lever whose length is equal to the distance between the fulcrum, or centre of motion at C, and the vertical line passing through the centre of gravity of the section ABCD; consequently, in the case of an equilibrium, we have

$$m = m' + \mathrm{M},$$

and this, by restoring the analytical values, becomes

$$\tfrac{1}{6}sd^3 = s\delta d(\tfrac{1}{2}b - \tfrac{1}{6}\delta) + \tfrac{1}{2}\mathrm{D}ls'(2b - c - e). \qquad (148).$$

209. This is the general equation which includes all the cases of rectilinear sloping embankments, but it has not yet obtained its ultimate form; for the value of l has still to be expressed in terms of the sectional dimensions, and in order to this, a separate investigation becomes necessary.



Thus, let ABCD be a vertical section of the dyke as before, and bisect the parallel sides AB and DC in the points m and n, and join mn; then, the straight line mn will pass through the centre of gravity of the figure ABCD.

Take g a point such, that mg is to ng, as $2\mathrm{DC}+\mathrm{AB}$ is to $\mathrm{BC}+2\mathrm{AD}$, and g will be the centre of gravity sought; through the points m and g, draw the straight lines mr and gs, respectively perpendicular to DC the base of the section, then is sc the length of the lever by which the weight of the dyke or embankment opposes the horizontal pressure of the fluid.

From the points A and B, draw the straight lines AK and BL, respectively perpendicular to DC; then it is manifest from the principles of geometry, that

$$rn = \tfrac{1}{2}(\mathrm{CL}-\mathrm{DK}),$$

and this, by restoring the symbols for CL and DK, becomes

$$rn = \tfrac{1}{2}(e-c).$$

But $mr = \mathrm{D}$; consequently, by the property of the right angled triangle, we have

$$mn^2 = mr^2 + nr^2;$$

or by restoring the analytical values, it is

$$mn^2 = \mathrm{D}^2 + \tfrac{1}{4}(e-c)^2;$$

therefore, by extracting the square root, we have

$$mn = \tfrac{1}{2}\sqrt{4\mathrm{D}^2+(e-c)^2}.$$

By the property of the centre of gravity, and according to the foregoing construction, the point g is determined in the following manner.

$$\begin{array}{rl} 2\mathrm{DC}+\mathrm{AB} = & 3b-c-e \\ \mathrm{DC}+2\mathrm{AB} = & 3b-2c-2e \\ \hline & 6b-3c-3e : \tfrac{1}{2}\sqrt{4\mathrm{D}^2+(e-c)^2} :: 3b-2c-2e : gn, \end{array}$$

from which, by reducing the analogy, we get

$$gn = \frac{(3b-2c-2e)\sqrt{4\mathrm{D}^2+(e-c)^2}}{6(2b-c-e)},$$

and by the property of similar triangles, it is

$$\tfrac{1}{2}\sqrt{4\mathrm{D}^2+(e-c)^2} : \tfrac{1}{2}(e-c) :: \frac{(3b-2c-2e)\sqrt{4\mathrm{D}^2+(e-c)^2}}{6(2b-c-e)} : sn,$$

wherefore, by reducing the analogy, we obtain

$$sn = \frac{(e - c)(3b - 2c - 2e)}{6(2b - c - e)}.$$

But by referring to the diagram, it will readily appear that $s\mathrm{c} = sn + n\mathrm{c}$; therefore, by addition, we obtain

$$l = \frac{3b(2b - c - e) + (e - c)(3b - 2c - 2e)}{6(2b - c - e)}.$$

Let this value of l be substituted instead of it in the equation marked (148), and we shall obtain

$$\tfrac{1}{6}sd^3 = s\delta d(\tfrac{1}{2}b - \tfrac{1}{6}\delta) + \frac{\mathrm{D}s'}{12}\{3b(2b - c - e) + (e - c)(3b - 2c - 2e)\},$$

and this being reduced to its simplest general form, becomes

$$sd^3 = d\delta s(3b - \delta) + 3b\,\mathrm{D}s'(b - c) + \mathrm{D}s'(c^2 - e^2). \quad (149).$$

210. The general equation in the form which it has now assumed, is very prolix and complicated; but its complication and prolixity, as we have before observed, are much increased by the introduction of the vertical pressure; if that element be omitted, the equation becomes

$$sd^3 = 3b\,\mathrm{D}s'(b - c) + \mathrm{D}s'(c^2 - e^2). \quad (150).$$

An expression sufficiently simple for every practical purpose; but it must be observed, that if e^2 be greater than c^2, the term in which it occurs will be subtractive.

We shall not attempt to express these equations in words, or to give practical rules for their reduction; the combinations are too complex, to admit of this being done in a neat and intelligible manner; it is necessary, however, to illustrate the subject by proper numerical examples, for which purpose, the following are proposed in this place.

211. Example 1. The water in a reservoir is 24 feet deep, and the wall which supports it is 30 feet in perpendicular height, the slope of the side next the water being one foot, and that of the opposite side one foot and a half; it is required to determine the transverse section of the wall or dyke, supposing it to be built of materials whose mean specific gravity is 2½, that of water being unity?

By contemplating the conditions of the question as here proposed, it will readily be observed, that the breadth of the section at the base, is the first thing to be determined from the equation; for since the quantity of the slopes, as well as the perpendicular height are given, the breadth of the dyke at top can easily be found, when the breadth at the foundation is known.

In the first place then, let us take into consideration the effect produced by means of the vertical pressure of the fluid; this will refer us to equation (149), but previously to the substitution of the several numerical quantities, it becomes necessary to assign the numerical value of δ, which is not expressed in the question, but is determinable from the perpendicular altitudes of the wall and the fluid, together with the slope of that side on which the fluid presses: thus,

$$30 : 24 :: 1 : \delta = \tfrac{4}{5} \text{ of a foot.}$$

Let therefore, the several given numbers replace their representatives in equation (149), and we shall have

$$24^3 = 24 \times \tfrac{4}{5} \times 2\tfrac{1}{2}(3b - \tfrac{4}{5}) + 3 \times 30 \times 2\tfrac{1}{2}(b-1)b - 30 \times 2\tfrac{1}{2}(1\tfrac{1}{2}^2 - 1^2),$$

in which expression, b is the unknown quantity.

If the several terms be expanded, collected, and arranged, according to the dimensions of the unknown quantity, we shall have

$$225b^2 - 81b = 13956.15;$$

complete the square, and we get

$$202500b^2 - 72900b \times 81^2 = 12567096,$$

and extracting the square root, it is

$$450b - 81 = \sqrt{12567096} = 3545 \text{ nearly};$$

therefore, by transposition and division, we get

$$b = 8.05 \text{ feet.}$$

Consequently, if from the breadth of the foundation as above determined, we subtract the sum of the slopes, the remainder will be the breadth of the dyke at the top; hence, the section can be delineated.

212. The above is the method of performing the operation, when the effect produced by the vertical pressure of the fluid is taken into consideration; but when that effect is omitted, the process is considerably shortened; for in the first place, there is no occasion to calculate the value of δ, that term not occurring in equation (150), and in the next place, there are fewer quantities to be substituted for; this greatly abbreviates the labour of reduction; but the equation is still of the same degree, and consequently, it must be resolved in the same manner.

Let the several given quantities remain as in the preceding case, and let them be respectively substituted in the equation (150), and we shall obtain

$$225b^2 - 225b = 13917.75;$$

if all the terms of this equation be divided by 225, the co-efficient of b^2, we shall get

$$b^2 - b = 61.856,$$

and this, by completing the square, becomes

$$b^2 - b + \tfrac{1}{4} = 62.106;$$

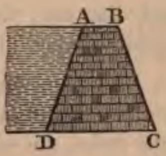
therefore, by extracting the square root and transposing, we have

$$b = 8.14 \text{ feet nearly.}$$

Corol. It therefore appears, that under the same circumstances, the computed breadth of the foundation differs very little, when the vertical pressure of the fluid is considered, from what it is when the pressure is omitted; and what is very remarkable, the difference, whatever it may amount to, leans to the side of safety and convenience in the case of the omission; it will therefore be sufficient in all cases of practice, to employ equation (150), but under certain circumstances of the data, it will admit of particular modifications.

213. When the slopes c and e are equal; that is, when the vertical transverse section of the dyke or embankment, is in the form of the frustum of an isosceles triangle, as represented by ABCD in the annexed diagram; then, the general equation (149), becomes transformed into

$$sd^3 = d\delta s(3b - \delta) + 3b\text{D}s'(b - c). \quad (151).$$



If the perpendicular height of the dyke, and the depth of the fluid, are equal to one another; that is, if the water is on a level with the top of the wall; then, $d = \text{D}$ and $\delta = c$, and the above equation becomes

$$sd^2 = cs(3b - c) + 3bs'(b - c). \quad (152).$$

Again, if we neglect the effect of vertical pressure, and express the specific gravity of water by unity, we get

$$3s'(b^2 - cb) = d^2. \quad (153).$$

And finally, if both sides of the equation be divided by the quantity $3s'$, we shall obtain

$$b^2 - cb = \frac{d^2}{3s'}. \quad (154).$$

The method of applying this equation is manifest, for we have only to substitute the given numerical values of c, d and s', and the value of b will become known by reducing the equation.

214. Example 2. The dyke or embankment which supports the water in a reservoir, is 20 feet in perpendicular height, and it slopes equally on both sides to the distance of 2 feet; what is the breadth of

the base, supposing the water to be on a level with the top of the wall, the specific gravity of the materials of which it is built being 1¾, that of water being unity?

Let these numbers be substituted for the respective symbols in the above equation, and we get

$$b^2 - 2b = 61.619,$$

complete the square, and it becomes

$$b^2 - 2b + 1 = 62.619,$$

from which, by evolution and transposition, we get

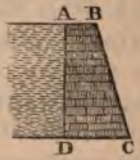
$$b = 8.913 \text{ feet nearly.}$$

Here then, the transverse section of the dyke is 8.913 feet across at the bottom, and consequently it is $8.913 - 4 = 4.913$ feet broad at the top; hence the delineation is very easily effected.

215. If the slope c should vanish; that is, if the side of the dyke on which the fluid presses be vertical, as represented by ABCD in the annexed diagram; then δ vanishes also, and the equation marked (149) becomes

$$s d^3 = 3 \text{D} s' b^2 - \text{D} s' e^2, \qquad (155).$$

where it is manifest there is no vertical pressure on the dyke, the whole effect of the fluid being exerted in the horizontal direction, tending to turn the wall about the remote extremity of its base.



When the perpendicular altitude of the wall or dyke, and the depth of the water are equal; then $d = \text{D}$, and admitting that the value of s, or the specific gravity of water is represented by unity, we obtain

$$d^2 = 3s' b^2 - s' e^2,$$

and this, by transposition and division, becomes

$$b^2 = \frac{d^2 + s' e^2}{3s'},$$

and lastly, by extracting the square root, we get

$$b = \sqrt{\frac{d^2 + s' e^2}{3s'}}. \qquad (156).$$

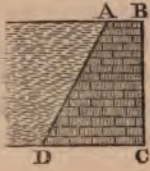
Let the slope of the dyke be two feet, its perpendicular altitude, or the depth of the fluid 20 feet, and the specific gravity of the material 1¾, as in the preceding example; then, by substitution, we obtain

$$b = \sqrt{\frac{400 +}{3 \times 1.75}} = 8.804 \text{ feet.}$$

Corol. The breadth at the base, as determined by this and the preceding equation, exhibits but a small difference, being in excess in the former case, by a quantity equal to 0.109 of a foot; but the breadth at the top in the latter case, exceeds that in the former, by a quantity equal to 1.891 feet; and the difference in the area of the section, is 17.82 feet: it is consequently more expensive to erect a dyke or embankment, with the side next the fluid perpendicular, than it is to erect one of equal stability with both sides inclined or sloping outwards.

216. If the slope e should vanish; that is, if the side of the dyke opposite to that on which the fluid presses, be perpendicular to the horizon, as represented by ABCD in the annexed diagram, then, the equation (149) becomes

$$sd^3 = d\delta s(3b - \delta) + 3\mathrm{D}s'(b^2 - cb) + \mathrm{D}c^2s'. \quad (157).$$



But when the effect of the vertical pressure of the fluid is omitted, we obtain

$$sd^3 = 3\mathrm{D}s'(b^2 - cb) + \mathrm{D}c^2s', \qquad (158).$$

and by supposing the altitude of the dyke, and the depth of the fluid to be equal (the specific gravity of the fluid being expressed by unity); then we have $d = \mathrm{D}$, and the foregoing equation becomes

$$d^2 = 3s'(b^2 - cb) + c^2s';$$

consequently, by transposition and division, we get

$$b^2 - cb = \frac{d^2 - c^2s'}{3s'}; \qquad (159).$$

from which equation, the value of b is easily determined.

Let the slope of that side of the dyke on which the fluid presses, be equal to 2 feet, and the perpendicular altitude of the dyke, or the depth of the fluid 20 feet, the specific gravity of the material being $1\frac{3}{4}$ as before; then by substitution, the foregoing equation becomes

$$b^2 - 2b = \frac{20^2 - 2^2 \times 1.75}{3 \times 1.75} = 74.8571;$$

by completing the square, we obtain

$$b^2 - 2b + 1 = 74.8571 + 1 = 75.8571;$$

consequently, by extracting the square root and transposing, we get

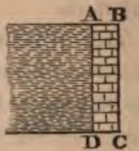
$$b = 9.709 \text{ feet}.$$

In this case, the dyke has less stability than it has when the perpendicular side is towards the water, as is manifest from its requiring a greater section, and consequently, a greater quantity of materials to resist the effort of the pressure which tends to overturn it.

The sectional area in the one case, is $7.804 \times 20 = 156.08$ square feet, and in the other, it is $8.709 \times 20 = 174.18$ square feet, being a difference of 18.1 square feet in favour of magnitude in the latter form, where the sloping side is adjacent to the fluid; and this being multiplied by the length of the dyke, will give the extra quantity of materials necessary for obtaining the same degree of stability.

217. If both the slopes c and e become evanescent; that is, if the section of the dyke be rectangular, having both its sides perpendicular to the horizon, as represented by ABCD in the annexed diagram; then, the general equation (149), becomes transformed into

$$s d^3 = 3 \mathrm{D} s' b^2. \qquad (160).$$



Then, by supposing the depth of the fluid, and the perpendicular altitude of the dyke to become equal, (the specific gravity of water being expressed by unity,) we have

$$3s' b^2 = d^2;$$

and this by division becomes

$$b^2 = \frac{d^2}{3s'};$$

consequently, if the square root of both sides of this equation be extracted, we shall have

$$b = \sqrt{\frac{d^2}{3s'}} = d\sqrt{\frac{1}{3s'}}. \qquad (161).$$

218. This is indeed a very simple form of the equation, applicable to the very important case of rectangular walls; it is however accurate, and corresponds in form with that investigated by other writers for the same purpose, and by different methods; the mode of its reduction is simply as follows.

> RULE. *Divide the specific gravity of the fluid to be supported, by three times the specific gravity of the dyke or embankment, and multiply the square root of the quotient by the perpendicular altitude of the dyke, for the required thickness.*

Let the perpendicular depth of the water, or the altitude of the dyke be equal to 20 feet, and the specific gravity of the materials of which it is built $1\frac{3}{4}$, as in the foregoing cases; then, by proceeding according to the rule, we have

$$b = 20\sqrt{\frac{1}{3 \times 1.75}} = 8.726 \text{ feet.}$$

219. There is still another case of very frequent occurrence that remains to be considered, viz. that in which the section is in the form of a right angled triangle, having its vertex on the same level with the surface of the fluid.

This case will also admit of two varieties, according as the perpendicular side of the dyke is, or is not in contact with the fluid; when it is in contact with it c vanishes, and since the section is in the form of a triangle, the breadth of the base b is equal to the remote slope e, and the vertical pressure of the fluid on the dyke is evanescent; consequently, the equation marked (149) becomes

$$sd^3 = 3\text{D}s'b^2 - \text{D}s'e^2; \qquad (162).$$

but by the nature of the problem e^2 is equal to b^2, and by the hypothesis of equal altitudes $d = \text{D}$; therefore, in the case of water, whose specific gravity is expressed by unity, we obtain

$$2s'b^2 = d^2;$$

and from this, by division, we get

$$b^2 = \frac{d^2}{2s'},$$

and finally, by extracting the square root, it is

$$b = d\sqrt{\frac{1}{2s'}}. \qquad (163).$$

220. This is also a very simple expression for the base of the section, and the rule for its reduction is simply as follows.

> Rule. *Divide the specific gravity of the incumbent fluid, by twice the specific gravity of the dyke or embankment, and multiply the perpendicular depth of the fluid by the square root of the quotient, for the required thickness of the dyke.*

Let the perpendicular altitude and the specific gravity of the wall, be 20 feet and $1\frac{3}{4}$ respectively, as in the foregoing cases, and we shall have

$$b = 20\sqrt{\frac{1}{2 \times 1.75}} = 10.68 \text{ feet.}$$

221. Lastly, if the fluid come in contact with, or press upon the hypothenuse of the triangle; then the slope e vanishes, and b and c are equal; consequently, equation (149) becomes

$$sd^3 = d\delta s(3b - \delta) + \text{D}s'b^2; \qquad (164).$$

and if the vertical pressure of the fluid be omitted, the first term on the right hand side of the equation vanishes, and consequently, we get

$$sd^3 = \text{D}s'b^2;$$

but $d = \text{D}$, and $s = 1$; therefore, we shall have

$$s' b^2 = d^2,$$

and finally, by division and evolution, we obtain

$$b = d\sqrt{\frac{1}{s'}} \qquad (165).$$

222. This equation is of a still simpler form than that which arises when the perpendicular side is towards the pressure, and the rule for its reduction is as follows.

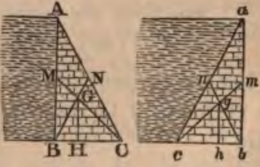
> RULE. *Divide the specific gravity of the fluid, by that of the dyke or embankment, and multiply the perpendicular altitude by the square root of the quotient, for the breadth of the base.*

Therefore, by taking the altitude and specific gravities hitherto employed, the rule will give

$$b = 20\sqrt{\frac{1}{1.75}} = 15.11 \text{ feet.}$$

223. This gives a thickness for the base of the section, exceeding the thickness in the former case by 4.33 feet; which seems to be a very great difference, when it is considered that both the form and the perpendicular altitude of the wall are the same in both cases; but the reason of the difference will become manifest from the following construction.

Let ABC and abc be two right angled triangles, equal to one another in every respect, but having their perpendiculars opposed in such a manner, that the water pressing in the same horizontal direction, is resisted by the perpendicular AB in the one case, and by the hypothenuse ac in the other.



Bisect the sides AB, AC in the point M and N, and ab, ac in the points m and n respectively; draw the lines CM and BN intersecting in G, and cm and bn intersecting in g; then are G and g, the centres of gravity of the respective triangles ABC and abc.

Demit the straight lines GH and gh, perpendicularly to BC and bc; then are HC and hb the levers, by which the weights of the sections, supposed to be concentrated in their respective centres of gravity, resist the horizontal pressure of the fluid which tends to turn them round the points C and b.

Now, according to the property of the centre of gravity, HC is equal to two thirds of BC, while *hb* is only one third of *bc*; but the horizontal pressure of the water is the same in both cases; it will therefore require the same mechanical energy to resist it; and since, by the conditions of the problem, the altitudes AB and *ab* are equal, it follows, that in order to produce an equilibrium, the product of the base of the triangle *abc*, into the length of the lever *hb*, must be increased in such a manner, that

$$\mathrm{BC} \times \mathrm{HC} = bc \times hb, \qquad (166).$$

and by converting this equation into an analogy, it becomes

$$\mathrm{BC} : bc :: hb : \mathrm{HC}.$$

We have seen, that by the construction and the property of the centre of gravity, the lever HC is equal to two thirds of BC, and *hb* equal to one third of *bc*; let therefore, $\frac{2}{3}$BC and $\frac{1}{3}bc$, be substituted for HC and *hb*, in the equation marked (166), and we shall obtain

$$\tfrac{2}{3}\mathrm{BC}^2 = \tfrac{1}{3}bc^2;$$

or dividing both terms by $\frac{1}{3}$, it becomes

$$2\mathrm{BC}^2 = bc^2,$$

and finally, by extracting the square root, it is

$$bc = \mathrm{BC}\sqrt{2}. \qquad (167).$$

Hence the reason for, and the nature of the increased breadth become obvious, the one being the side, and the other the diagonal of a square.

Now, we have found that the breadth of the dyke at the base, is equal to 10.68 feet, when the perpendicular side is in contact with the fluid; consequently, when the pressure is exerted on the hypothenuse, we have

$$b = 10.68 \times 1.4142 = 15.11 \text{ feet},$$

being the very same result as that which we obtained from the reduction of the equation (165).

224. What we have hitherto done, has reference to the case in which the obstacle yields to the pressure of the fluid, by turning upon the remote extremity of its base; we have therefore, in the next place, to investigate the conditions of equilibrium, when the obstacle is supposed to yield, by sliding along the horizontal plane on which it is erected.

Since the base of the dyke or wall is horizontal, it is manifest that the mass which it sustains, resists the horizontal pressure of the fluid, only by its adhesion to the base, and the resistance occasioned by friction.

Suppose therefore, that the resistances of adhesion and friction, are equal to n times the weight of the dyke, which we have represented by w; then we have

$$nw = \tfrac{1}{2}d^2 s;$$

but we have shown, in the investigation of the preceding case, that

$$w = \tfrac{1}{2}\mathrm{D}s'(2b - c - e);$$

consequently, by substitution, we obtain

$$d^2 s = \mathrm{D}ns'(2b - c - e). \qquad (168).$$

This is the equation of equilibrium, or that in which the resistance of the dyke is counterpoised by the horizontal pressure of the fluid, the effect of the vertical pressure not being considered; but in order to express the breadth of the base in terms of the other quantities, let both sides of the equation be divided by $\mathrm{D}ns'$, and it becomes

$$2b - c - e = \frac{d^2 s}{\mathrm{D}ns'};$$

consequently, by transposition and division, we obtain

$$b = \frac{d^2 s}{2\mathrm{D}ns'} + \tfrac{1}{2}(c + e); \qquad (169).$$

and finally, if the perpendicular depth of the fluid and the height of the dyke are equal, we shall have

$$b = \frac{ds}{2ns'} + \tfrac{1}{2}(c + e). \qquad (170).$$

225. In order therefore, to illustrate the reduction of the above equation by means of a numerical example, we must assume a value to the letter n, having some relation to the nature of the materials of which the resisting obstacle is constructed; now, it has been found by numerous experiments, that when rough and uneven bodies rub upon one another, or when a heavy body composed of hard and rough materials, is urged along a horizontal plane, the effect of the friction is equivalent to about one third of the weight of the body moved; or in other words, it requires about one third part of the force applied to overcome the effects of the friction; and moreover, in the case of a wall built of masonry, there is, in addition to the friction, the adhesion of the materials to the plane on which the wall is built.

If therefore, we consider the effect of adhesion to be equivalent to the effect of friction, it is manifest, that their conjoint effects will destroy about two thirds of the force applied; consequently, in the case of masonry, we may suppose that the value of n, is very nearly equal to $1\tfrac{1}{2}$, but for other materials it will vary according to the specific gravity or weight.

Having thus assigned a particular value to the letter n, we shall next proceed to illustrate the reduction of the equation; for which purpose, take the following example.

226. EXAMPLE 3. The vertical transverse section of the wall which supports the water in a reservoir, is 24 feet in perpendicular height; what is the thickness at the base of the wall, supposing the section to be in the form of the frustum of an isosceles triangle, the slope or inclination on each side, being equal to 2 feet, and the specific gravity of the material 1¾, that of water being expressed by unity?

Let the several numerical values here specified, be substituted instead of the respective symbols in the equation (170), and we shall obtain

$$b = \frac{24 \times 1}{2 \times \frac{3}{2} \times \frac{7}{4}} + 2 = 6.571 \text{ feet very nearly.}$$

The breadth of the section, or the thickness of the dyke at the bottom, being thus determined, the breadth or thickness at the top can easily be found, for we have

$$6.571 - 4 = 2.571 \text{ feet.}$$

127. If the slope c should vanish; that is, if the side of the dyke on which the water presses be perpendicular to the horizon; then, the equation (170), becomes

$$b = \frac{ds}{2ns'} + \tfrac{1}{2}e. \qquad (171).$$

And if the opposite slope e becomes evanescent, while the slope c remains; then we have

$$b = \frac{ds}{2ns'} + \tfrac{1}{2}c. \qquad (172).$$

But if both slopes vanish, or the section of the wall becomes rectangular; then, the equation (170) is

$$b = \frac{ds}{2ns'}. \qquad (173).$$

If therefore, the perpendicular altitude of the section, and the specific gravity of the materials of which the dyke is composed, remain as in the preceding example; then we shall have

$$b = \frac{24 \times 1}{2 \times \frac{3}{2} \times \frac{7}{4}} = 4.571 \text{ feet.}$$

228. When the section of the wall assumes the form of a right angled triangle; that is, when the slope c vanishes, and e becomes equal to the whole breadth b; then we have

$$b = \frac{ds}{ns'}. \qquad (174).$$

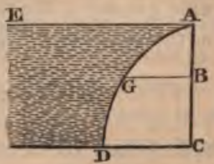
And exactly the same equation would arise, if the slope e remote from the fluid were to vanish, and the slope c adjacent to the fluid, become equal to b the whole breadth of the section; consequently, the thickness of a dyke in the case of a triangular section, whether the water presses on the perpendicular or hypothenuse of the triangle, is

$$b = \frac{24 \times 1}{\frac{3}{2} \times \frac{7}{4}} = 9.142 \text{ feet.}$$

In all the preceding cases, it is supposed that the section of the dyke or embankment is of such dimensions, as to oppose an equipoising resistance to the pressure of the fluid which it supports; but in the actual construction of all works of this nature, it becomes necessary, for the sake of safety, to enlarge the dimensions considerably beyond what theory assigns to them; but it does not belong to this place to determine the limits of the enlargement.

2. OF THE PRESSURE OF FLUIDS AGAINST EMBANKMENTS OF LOOSE MATERIALS.

229. The theory which we have established above, supposes that a perfect connection obtains between all the parts of the dyke or embankment which is opposed to the pressure of the fluid, so that any one portion of it cannot be displaced or overthrown, unless the whole be overthrown at the same time; the formulæ thence arising, are therefore, only applicable to dykes or embankments that are constructed of masonry; in those which are constructed of earth or other loose materials, and having the sides faced or fortified with stone, the same connection between the component portions of the wall does not exist, and consequently, although the several equations apply when the whole perpendicular height of the dyke is considered, yet the dyke will not resist equally at every part of the height, but is liable to be separated into horizontal sections.



In order therefore, to adapt our principles to this case also, it becomes necessary to trace out the steps of another investigation; for which purpose,

Let ACD represent a vertical section of the dyke or embankment, whose summit at A is on a level with the surface of the fluid AE.

Take any point G in the line AGD, and through the point G thus assumed, draw the horizontal ordinate GB, cutting the vertical axis AC in the point B: now, it is required to determine the nature of the curve AGD such, that each portion of the dyke, or of its section, as AGB, estimated from the vertex, may be equally capable of resisting the horizontal pressure of the fluid exerted against AG; or which is the same thing, that each portion may retain its stability and remain in equilibrio on its base GB; not separating from the lower portion GBCD, either by turning about the point C as a centre of motion, or by sliding in a horizontal direction along the base GB.

Put $x = \text{AB}$, the abscissa of the curve estimated from the vertex at A,
$y = \text{BG}$, the horizontal ordinate corresponding to the abscissa x,
$s =$ the specific gravity of the fluid, which endeavours to displace the dyke by pushing it along the line BG,
$s' =$ the specific gravity of the materials of which the dyke is constituted,
$m =$ the momentum of the horizontal pressure,
$m' =$ the momentum of the resistance offered by the dyke, and
$n =$ the number of times that the adhesion and friction of the dyke are equal to its weight.

Then we have already seen, equation (145), that the momentum of the horizontal pressure of the fluid as referred to the point C, is

$$m = \tfrac{1}{6} s d^3,$$

from which, by substituting x^3 instead of d^3, we obtain $m = \frac{1}{6} s x^3$; which equation indicates the momentum of pressure at the point D.

But the momentum of the resistance offered by the wall, that is, the momentum of the portion of the section represented by ABG, is

$$m' = \tfrac{1}{2} s' \int y^2 \dot{x};$$

and these momenta in the case of an equilibrium must be equal to one another; hence we have

$$\tfrac{1}{6} s x^3 = \tfrac{1}{2} s' \int y^2 \dot{x},$$

from which, by taking the fluxion, we shall obtain $\frac{1}{2} s x^2 \dot{x} = \frac{1}{2} s' y^2 \dot{x}$, or by suppressing the common factors, it becomes $s x^2 = s' y^2$; by extracting the square root of both terms, we get $x\sqrt{s} = y\sqrt{s'}$.

Now, when x becomes equal to d, the whole perpendicular depth of the fluid, or the altitude of the section; then y becomes equal to b, the thickness of the dyke, or the greatest breadth of the section;

consequently, if d and b be respectively substituted for x and y in the preceding equation, we shall have

$$d\sqrt{s}=b\sqrt{s'} \qquad (175).$$

This equation involves the conditions necessary for preventing the dyke from turning about the point B, and if the equation be resolved into an analogy, we shall have $b : d :: \sqrt{s} : \sqrt{s'}$.

Corol. From which we infer, that the section is in the form of a rectilinear triangle, whose base is to the perpendicular height, as the square root of the specific gravity of the fluid, is to the square root of the specific gravity of the wall or dyke.

230. Example. The perpendicular altitude of an embankment of earth is 20 feet; what must be the breadth of its base, so that each portion of it estimated from the vertex, shall resist the effort of the fluid, to turn it round the remote extremity of the base, with equal intensity; the water and the dyke having equal altitudes, and their specific gravities being 1 and 1.5 respectively?

Here we have given $d=20$ feet, $s=1$, and $s'=1.5$; consequently, by the preceding analogy, we have $\sqrt{1.5} : \sqrt{1} :: 20 : 16.33$ feet.

231. The conditions necessary for preventing the portion of the section ABG from sliding on its base, may be thus determined.

We have seen (art. 220), that the momentum of the horizontal pressure, to urge the section along its base, is $m=\frac{1}{2}sd^2$,

consequently, by substituting x^2 for d^2, we have $m=\frac{1}{2}sx^2$,

but the momentum of the section opposed to this, is $m'=ns'\int y\dot{x}$;

therefore in the case of an equilibrium, we have $\frac{1}{2}sx^2=ns'\int y\dot{x}$,

from which, by taking the fluxion, we obtain $sx\dot{x}=ns'y\dot{x}$,

and by casting out the common factor, we get $sx=ns'y$. (176).

From this equation, when converted into an analogy, we shall obtain

$$x : y :: ns' : s.$$

Which also indicates a rectilinear triangle, whose altitude is to the base, as n times the specific gravity of the embankment, is to the specific gravity of the fluid.

If the water presses against the perpendicular side of the wall, the curve bounding the other side, so that the strength of the wall may be every where proportional to the pressure which it sustains, must be a semi-cubical parabola, whose vertex is at the surface of the fluid, and convex towards the pressure.

232. We are now arrived at that particular division of our subject, which comprehends some of the most interesting and important departments of hydrodynamical science; it unfolds the principles of floatation, explains the method of weighing solid bodies in fluids, determines the relations of their specific gravities; and moreover, it investigates the laws of equilibrium, and assigns the conditions necessary for a state of perfect or imperfect stability. Every term in this enumeration conveys the idea of mechanical action.

Floating bodies, those which swim on the surface of a fluid, which is bulk for bulk heavier than the body afloat, are pressed downward by their own weight in a vertical line passing through their centre of gravity: and they are supported by the upward pressure of the fluid, which acts in a vertical line passing through the centre of gravity of the part which is under the water. When these lines are coincident, the equilibrium of floatation will be permanent. In the present instance we have merely to consider the principles of floatation as fluids exhibit the properties of the mechanical powers, as the lever or balance, the screw, &c. The pulley, in lowering a great weight or in lifting it up again, does no more than the ocean tide when it silently recedes and leaves dry, or majestically advances and without effort floats a stupendous ship. The lever or balance does no more than a canal lock effects, when it transfers from one level to another a heavy barge or vessel laden with ponderous commodities. And we behold too the ocean, like a vast screw or press, forcing down to its dark recesses vast masses, which in shipwrecks are submerged in its bosom, and which yet might be fashioned to be bulk for bulk much lighter than the devouring flood which has swallowed them up in its insatiable womb. The eternal and immutable laws of Nature, in all these cases, are most satisfactorily accounted for in the doctrine of fluid pressure and support; but this doctrine, like all the rudiments of human skill applied to natural phenomena, must depend on matters of fact, which can only be learned from observation and experiment, and which can generally and successfully be applied by the help of mathematical and philosophical investigations. This is the only scientific view we ought to take of all those truths that are denominated the phenomena of fluids, whose affections, from a series of concurring experiments, we undertake to expound; or assuming these as established principles that operate generally in the pressure and elasticity of fluids, we demonstrate them to be adequate to the production, not only of the particular effects adduced to prove their existence and power, but of all similar phenomēna. This is the only method by which to make the results of practical men available in scientific discussions, and on the other hand render these discussions the handmaids of genius in constructive mechanics. This is the province of the mathematician; and we shall in the sequel follow it very closely, in expounding the doctrine of floatation and the specific gravities of bodies, the laws of equilibrium, and the conditions necessary for a state of perfect or imperfect stability, &c.

CHAPTER IX.

OF FLOATATION, AND THE DETERMINATION OF THE SPECIFIC GRAVITIES OF BODIES IMMERSED IN FLUIDS.

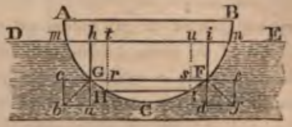
Of the several particulars with which we concluded the last chapter, we shall speak in order, beginning with the theory of floatation and the determination of the specific gravity of bodies, the leading principles of which are contained in the following proposition.

PROPOSITION III.

233. When a body floats, or when it is in a state of buoyancy on the surface of a fluid of greater specific gravity than itself:—

> *It is pressed upwards by a force, whose intensity is equivalent to the absolute weight of a quantity of the fluid, of which the magnitude is the same as that portion of the body below the plane of floatation.*[*]

Let ABC represent a vertical section of a solid body floating on a fluid, whose horizontal surface is DE, *mn* being the plane of floatation, and *m*C*n* the immersed portion of the floating body.



Take any two points G and H on the surface of the solid, indefinitely near to each other, and through the points G and H thus arbitrarily assumed, draw the straight lines GF and HI, respectively parallel to DE the surface of the fluid, and meeting the opposite sides of the solid in the points F and I, so that each point in either of the intercepted portions GH and FI, may be considered as being at the same perpendicular depth *h*G or *i*F below the horizontal surface of the fluid.

At H and I erect the perpendiculars H*r* and I*s*, which produce to *t* and *u*, and through the points G and F, draw the straight lines G*b* and F*f*, respectively perpendicular to the surface of the solid in the points G and F; make $\mathrm{G}b$ and $\mathrm{F}f$ each equal to $h\mathrm{G}$ or $i\mathrm{F}$, the perpendicular depth of the points G and F below the surface DE; then, according to the principles which we have propounded and demonstrated in the first proposition and its subordinate inferences, the perpendicular pressures upon the indefinitely small portions of the body GH and FI, may be expressed as follows, viz.

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[*] *The Plane of Floatation* is the imaginary plane, in which the floating solid is supposed to be intersected by the horizontal surface of the fluid.

$$p = s \times \mathrm{GH} \times \mathrm{G}b, \text{ and } p' = s \times \mathrm{FI} \times \mathrm{F}f,$$

where s denotes the specific gravity of the fluid, and p, p' the respective pressures exerted by it perpendicularly to GH and FI, any indefinitely small portions of the floating body.

But it is manifest from the resolution of forces, that the pressures of the fluid in the directions $b\mathrm{G}$ and $f\mathrm{F}$, may each be decomposed into two other pressures, the one vertical and the other horizontal; for by completing the rectangular parallelograms $\mathrm{G}abc$ and $\mathrm{F}dfe$, it is obvious that the pressures in the directions $a\mathrm{G}$, $c\mathrm{G}$ and $d\mathrm{F}$, $e\mathrm{F}$ are, when taken two and two, respectively equivalent to the pressures in the directions $b\mathrm{G}$ and $f\mathrm{F}$.

Now, the horizontal pressures $c\mathrm{D}$ and $e\mathrm{F}$, by construction are equal to one another, and they operate in contrary directions; consequently they destroy each other's effects, and the upward vertical pressures on the solid at the points G and F, are respectively indicated by the straight lines $a\mathrm{G}$ and $d\mathrm{F}$ drawn into the specific gravity of the fluid; therefore, the whole vertical pressures on the indefinitely small portions GH and FI, are as follows, viz.

$$p = s \times \mathrm{GH} \times a\mathrm{G}, \text{ and } p' = s \times \mathrm{FI} \times d\mathrm{F},$$

where p and p', instead of indicating the perpendicular pressures as formerly, are now considered in reference to the vertical pressures.

Since the parallel straight lines GF and HI are indefinitely near to one another, the lines GH and FI may be assumed as nearly straight, and consequently, the elementary triangles $\mathrm{GH}r$ and $\mathrm{FI}s$ are respectively similar to the triangles $\mathrm{GB}a$ and $\mathrm{F}fd$; therefore, by the property of similar triangles, we have

$$\mathrm{G}b : \mathrm{G}a :: \mathrm{GH} : \mathrm{G}r, \text{ and } \mathrm{F}f : \mathrm{F}d :: \mathrm{FI} : \mathrm{F}s;$$

and from these analogies, by equating the products of the extreme and mean terms, we obtain

$$\mathrm{G}b \times \mathrm{G}r = \mathrm{G}a \times \mathrm{GH}, \text{ and } \mathrm{F}f \times \mathrm{F}s = \mathrm{F}d \times \mathrm{FI}.$$

Let therefore, the products $\mathrm{G}b \times \mathrm{G}r$ and $\mathrm{F}f \times \mathrm{F}s$ be substituted instead of $\mathrm{GH} \times a\mathrm{G}$ and $\mathrm{FI} \times d\mathrm{F}$ in the above values of p and p', and we shall have

$$p = s \times \mathrm{G}b \times \mathrm{G}r, \text{ and } p' = \qquad f \times \mathrm{F}s.$$

Now, these pressures are manifestly equal to the weights of the columns G t and F u considered as fluid, and since the same may be demonstrated with respect to every other portion of the immersed surface, we therefore conclude, that the whole pressure upwards, is equal to the sum of the weights of all the columns G t, F u, &c.; that is, to the weight of a quantity of the fluid equal in magnitude to the immersed part of the body; hence the truth of the proposition is manifest.

Corol. From the principles demonstrated above, it follows, that when a solid body floating on the surface of a fluid is in a state of quiescence:—

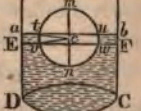
> *The pressure downwards is equal to the buoyant effort; that is, the weight of the floating body, is equal to the weight of a quantity of the fluid, whose magnitude is the same as that portion of the solid, which falls below the plane of floatation.*

PROBLEM XXXI.

234. A cylindrical vessel of a given diameter, is filled to a certain height with a fluid of known specific gravity, and a spherical body of a given magnitude and substance is placed in it:—

> *It is required to determine how high the fluid will rise in consequence of the immersion of the spherical segment which falls below the plane of floatation.*

Let ABCD represent a vertical section passing along the axis of a cylindrical vessel, filled with an incompressible and non-elastic fluid to the height ED, EF being the surface of the fluid before the sphere whose diameter is mn, is placed in it, and ab the surface after the immersion of the segment tnu, the liquid rising to the height AD.



Then it is manifest from the nature of the problem, that the spherical segment $tvnwu$, together with the quantity of fluid in the vessel, must be equal to the capacity of the cylinder whose diameter is DC, and perpendicular altitude aD; for the fluid rises in consequence of the immersion of the segment, and fills the spaces atvE and buwF all around the vessel; we have therefore to calculate the spherical segment *tvnwu*, and the cylinders EFCD and *ab*CD, for which purpose,

Put $d =$ ED, the height to which the vessel is originally filled with the fluid,
$\delta =$ DC, the diameter of the cylindrical vessel,
$r = ct$, or cv, the radius of the sphere,
$s =$ the specific gravity of the fluid in the vessel,
$s' =$ the specific gravity of the floating body, and
$x = a$D, the height to which the fluid rises on the immersion of the spheric segment.

Then, since by the principles of mensuration, the solid content or capacity of a sphere, is equal to two thirds of that of its circumscribing cylinder, it follows, that the capacity of the sphere *mvnw*, is expressed by

$$3.1416r^2 \times 2r \times \tfrac{2}{3} = 4.1888r^3;$$

but as we have elsewhere demonstrated, that the magnitudes of bodies are inversely as their specific gravities; consequently, the magnitude of the part immersed, is determined by the following analogy, viz.

$$s : s' : : 4.1888r^3 : \frac{4.1888r^3 s'}{s}.$$

Now, as we have already observed, the quantity of fluid in the vessel at first, is

$$.7854 \times \delta^2 \times d = .7854d\delta^2,$$

and the capacity of the cylinder formed by the fluid and the spherical segment, is

$$.7854 \times \delta^2 \times x = .7854\delta^2 x;$$

consequently, by addition, we shall have

$$.7854\,\delta^2 x = .7854d\delta^2 + \frac{4.1888r^3 s'}{s};$$

and therefore, if all the terms of this equation be divided by the quantity $.7854\delta^2$, we shall obtain

$$x = d + \frac{16r^3 s'}{3\delta^2 s}. \qquad (177).$$

Or if the height to which the vessel is originally filled, be subtracted from both sides of the above expression, the increase of height in consequence of the immersion of the spheric segment, becomes

$$x - d = x' = \frac{16r^3 s'}{3\delta^2 s}, \qquad (178).$$

where $x' = a$E the increase of height.

235. Either of these equations will resolve the problem, but the latter form is the most convenient for a verbal enunciation, and the practical rule which it supplies is as follows.

> RULE. *Multiply sixteen times the specific gravity of the sphere, by the cube or third power of its radius; then, divide the product by three times the specific gravity of the fluid, drawn into the square of the cylinder's diameter, and the quotient will give the increase of height, in consequence of the immersion of the spheric segment.*

236. EXAMPLE. A cylindrical vessel whose diameter is 8 inches, is filled with water to the height of 10 inches; how much higher will the water rise, and what will be its whole weight, when a globe of alder of 6 inches diameter is dropped into the vessel; the specific gravity of alder being equal to .8, when that of water is expressed by unity?

Here, by operating according to the above rule, we get

$$16r^3s' = 16\times3\times3\times3\times.8 = 345.6,$$

and in like manner we have

$$3\delta^2 s = 3\times8\times8\times1 = 192;$$

consequently, by division, we obtain

$$x' = \frac{16r^3s'}{3\delta^2 s} = \frac{345.6}{192} = 1.8 \text{ inches, and the whole height is } 11.8 \text{ inches.}$$

237. If the specific gravity of the globe, and that of the fluid in which it is placed, are equal to one another, then equation (178) becomes

$$x' = \frac{16r^3}{3\delta^2}. \qquad (179).$$

In this case it is manifest, that the sphere is wholly immersed in the fluid; consequently, the increase of height will be equal to the altitude of a cylinder, whose diameter is δ, and whose capacity is equal to that of the immersed body; hence, the method of computation is obvious; but the practical rule deduced from the equation for this purpose, may be expressed in the following manner.

> RULE. *Divide sixteen times the cube or third power of the radius of the sphere, by three times the square of the cylinder's diameter, and the quotient will give the increased height of the fluid.*

238. EXAMPLE. A cylindrical vessel whose diameter is 12 inches, is filled with fluid to the height of 6 inches; to what height will

the fluid ascend when a sphere of 4 inches diameter is placed in it, the specific gravities of the fluid and the sphere being equal to one another?

In this example there are given $\delta = 6$ inches and $r = 2$ inches; therefore, by operating as directed in the rule, we shall have

$$16r^3 = 16 \times 2^3 = 128, \text{ the dividend,}$$

and in like manner for the divisor, we get

$$3\delta^2 = 3 \times 12^2 = 432, \text{ the divisor;}$$

consequently, by division, we obtain

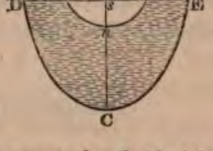
$$x' = \tfrac{128}{432} = 0.\dot{2}96\dot{2} \text{ of an inch.}$$

Hence it appears, that the height of the fluid in the vessel, is increased by a quantity equal to $0.\dot{2}96\dot{2}$ of an inch, in consequence of the immersion, and the whole height to which it rises, is $6.\dot{2}96\dot{2}$ inches.

PROBLEM XXXII.

239. A vessel in the form of a paraboloid, is placed with its vertex downwards and its base parallel to the horizon; now, supposing the vessel to be filled to the n^{th} part of its capacity with a fluid of known specific gravity, and let a spherical body of a given size and substance be placed in it:—

> *It is required to ascertain the height to which the fluid will rise, in consequence of the immersion of the spherical segment.*



Let ABC represent a vertical section passing along the axis of the vessel, whose form is that of a paraboloid, generated by the revolution of the common parabola; and suppose the vessel to be filled with an incompressible and non-elastic fluid to the height sc, DE being its horizontal surface when in a state of quiescence, before the sphere whose diameter is mn is placed in it; then will ab be the surface or the plane of floatation after the immersion of the segment rnw, the fluid rising to the height ts all around the spherical body.

Now, it is obvious from the nature of the problem, that the solidity of the spherical segment rnw, together with the quantity of fluid in the vessel, is equal to the magnitude of the paraboloid acb, whose base is ab and axis $\mathrm{c}t$; therefore, in order to calculate the solidity of the segment, and that of the paraboloids DCE and $a\mathrm{c}b$,

Put $d = m\mathrm{c}$, the whole axis or height of the paraboloid,
$p =$ the parameter or latus rectum of the axis,
$r = cr$, cm or cn, the radius of the sphere,
$s =$ the specific gravity of the fluid in the vessel,
$s' =$ the specific gravity of the floating body, and
$x = t\mathrm{c}$, the whole height to which the fluid ascends, n being the part originally filled.

By the principles of solid mensuration, the capacity or solidity of a sphere, is equivalent to two thirds of that of its circumscribing cylinder; consequently, the capacity of the floating sphere, is

$$3.1416r^2 \times 2r \times \tfrac{2}{3} = 4.1888r^3;$$

now, we have demonstrated in another place, that the magnitudes of bodies, are inversely as their respective gravities; hence we have for that portion actually immersed,

$$s : s' :: 4.1888r^3 : \frac{4.1888r^3 s'}{s}.$$

Again, by the principles of mensuration, the solidity of a paraboloid is equal to one half the solidity of its circumscribing cylinder, and by the property of the parabola, we have

$$m\mathrm{A}^2 = pd;$$

therefore, the capacity of the paraboloidal vessel, is

$$3.1416 \times pd \times \tfrac{1}{2}d = 1.5708pd^2,$$

and consequently, the quantity of fluid in it is expressed by

$$3.1416 \times pd \times \tfrac{1}{2}d \times \tfrac{1}{n} = \frac{1.5708pd^2}{n}.$$

But the capacity, or the solid content of the paraboloid $a\mathrm{c}b$, whose axis is $t\mathrm{c}$, becomes

$$3.1416 \times px \times \tfrac{1}{2}x = 1.5708px^2;$$

consequently, by addition and comparison, we have

$$1.5708px^2 = \frac{4.1888r^3 s'}{s} + \frac{1.5708pd^2}{n},$$

and dividing all the terms by 1.5708, we get

$$px^2 = \frac{8r^3 s'}{3s} + \frac{pd^2}{n},$$

and again, if all the terms be divided by p the parameter of the parabola, and the square root be extracted from both sides of the equation, we shall have

$$x=\sqrt{\frac{8r^3s'}{3sp}+\frac{d^2}{n}}.$$

But because the parameter of a parabola is a third proportional to any abscissa and its ordinate; it follows, that if b denote the base A B of the paraboloid, of which the axis is d, we shall have

$$p=\frac{b^2}{4d};$$

let this value of the parameter be substituted instead of it in the above equation, and we shall obtain

$$x=\sqrt{\frac{32dr^3s'}{3b^2s}+\frac{d^2}{n}}. \qquad (180).$$

240. The following practical rule supplied by this equation, will serve to direct the reader to the method of its reduction.

> RULE. *Multiply thirty-two times the axis of the paraboloid, by the cube or third power of the radius of the sphere drawn into its specific gravity; then, divide the product by three times the square of the base of the vessel multiplied by the specific gravity of the fluid, and to the quotient, add the square of the axis or depth of the vessel, divided by the number, which expresses what part of it is occupied by the fluid; then, the square root of the sum, will give the height to which the fluid rises after the immersion of the spheric segment.*

241. EXAMPLE. The axis of a vessel in the form of a paraboloid is 27 inches, and the diameter of its mouth is 18 inches; now, supposing that the vessel is one fifth full of water, into which is dropped a sphere of hazel whose diameter is 8 inches; to what point of the axis will the fluid ascend, the specific gravity of hazel being 0.6, when that of water is expressed by unity?

By proceeding according to the rule, we get

$32\times27\times4\times4\times4\times.6=33177.6$, the dividend,

and in like manner, for the divisor, we have

$3\times18\times18\times1=972$, the divisor;

consequently, by division, we obtain

$$\frac{33177.6}{972}=34.13.$$

This is the value of the first term under the radical sign, in the expression for x equation (180), and the value of the second term, is

$$\frac{27^2}{5}=145.8;$$

therefore, by addition and evolution, we obtain

$$x=\sqrt{179.9\dot{3}}=13.414 \text{ inches nearly.}$$

242. If the specific gravity of the ball, and that of the fluid in which it is placed, be equal to one another, then equation (180) becomes

$$x=\sqrt{\frac{32dr^3}{3b^2}+\frac{d^2}{n}},$$

and by reducing the fractions under the radical sign or vinculum to a common denominator, we obtain

$$x=\sqrt{\frac{d(32nr^3+3b^2d)}{3b^2n}}. \qquad (181).$$

243. The practical method of reducing the above equation, is expressed in words at full length in the following rule.

> RULE. *Multiply the cube or third power of the radius of the sphere, by thirty two times the number which indicates what part of the vessel is occupied by the fluid, and to the product add three times the axis of the vessel drawn into the square of its diameter; then, divide the sum by three times the square of the vessel's diameter, drawn into the number which denotes what part of it the fluid occupies; multiply the quotient by the axis of the vessel, and extract the square root of the product, for the height to which the fluid rises.*

244. EXAMPLE. Let the dimensions of the vessel and the immersed body, remain as in the preceding example, the vessel containing also the same quantity of fluid; to what height on the axis will the fluid ascend, supposing its specific gravity to be the same as that of the immersed body?

Here, by operating as directed by the rule, we get

$$32nr^3=32\times5\times4\times4\times4=320\times32=10240, \text{ and}$$

$$3b^2d=3\times18\times18\times27=972\times27=26244;$$

consequently, the sum of the parenthetical terms is

$$32nr^3+3b^2d=10240+26244=36484,$$

and for the denominator of the fraction, we have

$$3b^2n=3\times18\times18\times5=324\times15=4860;$$

consequently, by division, we obtain

$$\frac{32nr^3+3b^2d}{3b^2n}=\frac{36484}{4860}=7.507;$$

therefore, by multiplication we shall have

$$7.507\times27=202.689,$$

and finally, by evolution, it is

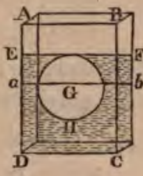
$$x=\sqrt{202.689}=14.23 \text{ inches.}$$

Corol. Hence it appears, that when the specific gravities of the fluid and the immersed body, are equal to one another, the fluid rises in the vessel to the height of 14.23 inches; but when the specific gravities are to each other as 1 : 0.6, it rises only to 13.414; the reason of the difference, however, is manifest, for in the case of equal specific gravities, the spherical body is wholly immersed; but when the specific gravities are unequal, only a part of the body falls below the plane of floatation. From the above we deduce the following inferences.

245. Inference 1. If a homogeneous body be immersed in a fluid of the same density with itself:—

> *It will remain at rest, or in a state of quiescence, in all places and in all positions.*

Let ABCD represent a vessel, filled with an incompressible and non-elastic fluid to the height aD, and let G be a homogeneous body, of the same density or specific gravity as the fluid.



Now, it is manifest, that when the body G is put into the vessel and left to itself, it will by reason of its own weight, sink below ab the original surface, and raise the fluid to the height ED, where the body will be entirely under the fluid, and the whole mass in a state of equilibrium with the surface at EF.

Then it is evident, that the body being of the same density as the fluid in which it is placed, it will press the fluid under it, just as much as the same quantity of the fluid would do if put in its stead, and consequently, the pressure exerted by the solid, together with that of the superincumbent fluid, presses downwards with the same energy, as if it were a column of fluid of equal depth.

Therefore, the pressure of the body against the fluid at H, is equal to the pressure of the fluid against the body there; consequently, these two pressures are equal and opposite to one another, and must therefore be in a state of equilibrium, in which case, the body will remain at rest.

Hence, the truth of the inference is manifest with respect to a vertical pressure; but it is equally true in reference to a motion horizontally and obliquely; for the horizontal pressures are obviously equal to one another, and they are in opposite directions; therefore, they are in equilibrio with one another, and no motion can take place.

And again, with regard to the oblique pressure, it is evidently compounded of a vertical and horizontal one; but we have just demonstrated that these are equal and opposite; consequently, the body can have no oblique motion, it must therefore remain at rest in any place and in any position.

If the specific gravity of the immersed body be greater than that of the fluid, the pressure downwards will exceed the pressure upwards; consequently, the weight of the body will overcome the resistance of the fluid under it, and it will therefore sink to the bottom.

But if the specific gravity of the body be less than that of the fluid, the pressure upwards will exceed the pressure downwards; therefore, the buoyant principle will overcome the weight of the solid, and it will rise to the surface of the fluid.

246. INF. 2. If a solid body be immersed in a fluid, and the whole mass be in a state of equilibrium:—

> *The pressure upwards against the base of the body, is equal to the weight of a quantity of fluid of equal magnitude, together with the weight of the superincumbent fluid.*

247. INF. 3. If a solid body be placed in a fluid of greater or less specific gravity than itself:—

> *The difference between the pressures downwards and upwards is equal to the difference between the weight of the solid and that of an equal bulk of the fluid.*

248. INF. 4. Heavy bodies when placed in fluids have a twofold gravity, the one true and absolute, the other apparent or relative.

> *Absolute gravity is the force with which bodies tend downwards.*

By reason of this force, all sorts of fluid bodies gravitate in their proper places, and their several weights, when taken conjointly, compose the weight of the whole; for the whole is possessed of weight, as may be experienced in vessels full of liquor.

Apparent or relative gravity, is the excess of the gravity of the body above that of the fluid in which it is placed.

By this sort of gravity, fluids do not gravitate in their proper places; that is, they do not preponderate; but opposing one another's descent, they retain their positions as if they were possessed of no weight.

249. INF. 5. If a heavy irregular heterogeneous body descends in a fluid, or if it moves in any direction, and a straight line be drawn, connecting the centres of magnitude and gravity of the body:—

It will so dispose itself as to move in that line, the centre of gravity preceding the centre of magnitude.

This is a manifest and a beautiful fact; for the centre of gravity being surrounded by more matter and less surface than the centre of magnitude, it will meet with less resistance from the fluid; consequently, the body will so arrange itself, as to move in the line of direction with its centre of gravity foremost.

250. What has been here adverted to, in regard to bodies of greater density or specific gravity sinking in a fluid, must only be understood to apply to such as are solid; for if a body be hollow, it may swim in a fluid of less specific gravity than that which is due to the substance of which the body is composed; but if the hollows or cavities are filled with the fluid, the body will then descend to the bottom.

Again, if bodies of greater specific gravity than the fluid in which they are placed, be reduced to extremely small particles, they may also be suspended in the fluid; but the principle or force by which this is effected, does not belong to hydrodynamics.

PROPOSITION IV.

251. If a solid homogeneous body, be placed in a fluid of greater or less specific gravity than itself:—

It will ascend or descend with a force, which is equivalent to the difference between its own weight, and that of an equal bulk of the fluid.

The principle announced in this proposition is almost self-evident, yet nevertheless, it may be demonstrated in the following manner.

Put $m =$ the common magnitude of the body and the fluid,
$w' =$ the weight of the solid body,
$s' =$ its specific gravity,

$w =$ the weight of an equal quantity of the fluid,
$s =$ its specific gravity, and
$f =$ the force with which the body ascends or descends in the fluid.

Then, because as we have elsewhere demonstrated, the absolute weights of bodies, are as their magnitudes and specific gravities; it follows, that

$$w = ms, \text{ and } w' = ms';$$

but according to the third inference preceding, the difference between the pressures downwards and upwards :—

> *Is equal to the difference between the weight of the solid body, and that of an equal bulk of the fluid.*

But the difference between the upward and downward pressures, is equivalent to the force of ascent and descent; consequently, we have

$$f = w \backsim w' = ms \backsim ms',$$

and this, by collecting the terms, becomes

$$f = m(s \backsim s'). \qquad (182).$$

If, therefore, the specific gravity of the solid be less than that of the fluid, the force of ascent will be

$$f = m(s - s');$$

but when the specific gravity of the solid exceeds that of the fluid, the force of descent becomes

$$f = m(s' - s),$$

and when the specific gravities are equal to one another, the force of ascent and descent vanishes, in which case, the body will remain at rest, in whatsoever position it may be placed; this agrees with what we have already stated in the first inference to Problem 32.

From the above proposition and its subordinate formulæ, the following inferences may be deduced.

252. INF. 1. When a solid body is immersed, or suspended in a fluid of equal or of different specific gravity :—

> *It loses the weight of an equal magnitude of the fluid in which it is placed.*

This is obvious, for when the specific gravities are equal, the body loses the whole of its weight; and therefore, it neither endeavours to ascend nor descend; but when the specific gravities are unequal, the body only endeavours to ascend or descend, by the difference between its own weight and that of an equal bulk of the fluid, and has therefore lost the weight of as much fluid.

253. INF. 2. When a solid body is immersed, or suspended in a fluid of the same, or of different specific gravity :—

> *It loses the whole or a part of its weight, according as it is totally or partially immersed, and the fluid gains the weight which the body loses.*

This is manifest, for the sum of the weights of the body and the fluid must be the same, both before and after the immersion.

254. INF. 3. If bodies of equal magnitude are placed in the same fluid, whatever may be their specific gravities :—

> *They lose equal weight, and unequal bodies lose weights that are proportional to their magnitudes.*

255. INF. 4. If the same body be immersed, or suspended in fluids of different specific gravities :—

> *The weights lost by the body are as the densities or specific gravities of the fluids.*

256. INF. 5. When two bodies of unequal magnitude are in equilibrio with one and the same fluid :—

> *They will lose their equilibrium, if they be transferred to another fluid of different density.*

257. INF. 6. When a body ascends or descends, in a fluid of greater or less specific gravity than itself :—

> *The force which accelerates its ascent or descent, is equal to the quotient that arises, when the difference between the weight of the body, and that of an equal bulk of the fluid, is divided by the common magnitude.*

It consequently follows, that when the solid is entirely immersed in the fluid, the force which urges its ascent or descent is constant; in which case, the motion upwards or downwards must be uniformly accelerated, if it be not disturbed by the resistance of the medium in which it moves.

258. If the solid is specifically heavier than the fluid, it will tend downwards, and press the bottom of the vessel, with a force which is equivalent to the excess of its weight above an equal bulk of the fluid; and this is what we understand by the *relative gravity* of the body in the fluid.

But if the body be specifically lighter than the fluid in which it is placed, it seems to lose a greater weight than it actually possesses, and consequently, it tends upwards, with a force equal to the difference between its own weight and that of an equal bulk of the fluid; and continuing to ascend, it will attain a position in which its weight is equal to that of a quantity of the fluid of the same magnitude as the part immersed; and this is what we understand by the *relative levity* of the body in the fluid.

259. These inferences being admitted, we shall now proceed to exemplify the general formula resulting from our proposition, viz. that in which we have

$$f = m(s \backsim s').$$

The practical rule for reducing this equation, may be expressed in general terms in the following manner.

> RULE. *Multiply the common magnitude of the body and the fluid, by the difference of their specific gravities, and the product will be the force of ascent or descent, according as the specific gravity of the body is less or greater than that of the fluid in which it is placed.*

260. EXAMPLE. A mass of dry oak, whose magnitude is equal to 7 cubic feet, and specific gravity equal to 0.8 (that of water being unity), is plunged into a vessel of fluid, whose specific gravity is 0.932; with what force will it ascend?

Here, according to the rule, we have

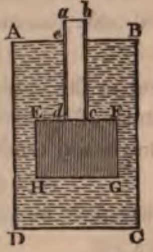
$f = m(s - s') = 7(.932 - .8) = .924$ of a cubic foot of the body whose specific gravity is 0.932; consequently, for the force in lbs. avoirdupois, we have 1 : 62.5 : : 924 : 57.75 lbs.

PROBLEM XXXIII.

261. In a vessel filled with an incompressible and non-elastic fluid, is placed a hollow cylinder, which we shall consider as being perfectly void of gravity or weight; to the bottom of which, a cylindrical body of a given magnitude, and whose specific gravity is greater than that of the fluid, is so closely fitted that no fluid can enter:—

> *It is required to determine, how far below the surface of the fluid the body will descend, before the tendency downwards, and the pressure upwards, are in equilibrio with one another.*

Let ABCD in the annexed diagram, be a vertical section of the vessel containing the fluid, *abcd* a corresponding section of the hollow cylinder, and EFGH that of the attached or cylindrical body.



Now, it is manifest, that in consequence of the connection between the hollow cylinder and the attached body, the downward pressure of the fluid can have no effect upon that portion of the upper surface of the body whose diameter is *dc*; and because the hollow cylinder *abcd*, is supposed to be without weight, it can have no influence on the downward tendency of the body EFGH: an equilibrium will therefore obtain, when the downward pressure on the surface E*d*, *c*F, together with the weight of the body, is equal to the upward pressure on the bottom HG.

Put $d = dc$, the diameter of the hollow cylinder destitute of weight,
$\delta = \text{EF}$, the diameter of the attached body,
$l = \text{EH}$, its perpendicular length, or vertical altitude,
$a =$ the area of the end of the hollow cylinder,
$\text{A} =$ that of the attached cylindrical body,
$s =$ its specific gravity, greater than that of the fluid,
$s' =$ the specific gravity of the fluid,
$w =$ the weight of the body,
$p =$ the pressure on its upper surface,
$p' =$ the pressure on its base, and
$x = ed$, the distance below AB, the upper surface of the fluid.

Then, by the mensuration of surfaces, the area of the lower extremity of the hollow cylinder *abcd*, becomes $a = .7854d^2$,

and that of the base of the attached body, is $\text{A} = .7854\delta^2$;

and the difference of these, or the quantity of the upper surface of the body, which is exposed to the downward pressure of the fluid, is

$$\text{A} - a = .7854\,(\delta^2 - d^2);$$

consequently, the downward pressure becomes $p = .7854\,(\delta^2 - d^2)\,s'x$; but the absolute weight of the body is expressed by its magnitude or solidity, drawn into its specific gravity; consequently the expression for the weight of the attached body becomes $w = \text{A}ls = .7854\delta^2 ls$;

therefore, we have for the whole tendency downwards,

$$p + w = .7854\,(\delta^2 - d^2)\,s'x + .7854\delta^2 ls,$$

and from this, by collecting the terms, we obtain

$$p + w = .7854\,\{(\delta^2 - d^2)\,s'x + \delta^2 ls\},$$

Now, the pressure upwards on the bottom of the attached cylindrical body, according to the principle of the first proposition, is

$$p' = .7854\,\delta^2 s'(l + x);$$

but in the case of an equilibrium, or when the body has attained a state of quiescence, the pressure upwards is exactly equal to the downward tendency; consequently, by comparison, we have

$$.7854\,\delta^2 s'(l + x) = .7854\,\{(\delta^2 - d^2)\,s'x + \delta^2 ls\};$$

therefore, by suppressing the common factor .7854, and transposing, we get

$$\delta^2 s'(l + x) - (\delta^2 - d^2)\,s'x = \delta^2 ls, \text{ and this,}$$

by expanding and collecting the terms, becomes $d^2 s'x = \delta^2 l(s - s')$,

and finally, by division, we obtain $x = \dfrac{\delta^2 l(s - s')}{d^2 s'}$. (183).

262. The following practical rule, drawn out in words at length, will serve for the reduction of the equation.

> Rule. *Multiply the difference between the greater and less specific gravities, by the square of the diameter of the attached body drawn into its length; then divide the product by the square of the diameter of the hollow cylinder, drawn into the specific gravity of the fluid, and the quotient will be the distance below the surface of the fluid at which the body rests.*

263. Example. A cylinder of lignum vitæ, whose diameter is 8 inches, length 36 inches, and specific gravity 1.327, is attached to the lower end of a hollow tube, whose diameter is 3 inches, in such a manner that no fluid can enter; now, supposing the body, and the hollow cylinder to which it is attached, to be placed in a vessel full of water, it is required to determine, at what distance below the surface of the fluid the body will become quiescent?

Here, by operating according to the rule, we obtain

$$x = \frac{8^2 \times 36(1.327 - 1.000)}{3 \times 3 \times 1} = 83.707 \text{ inches.}$$

Hence it appears, that a cylinder of lignum vitæ of the proposed dimensions, will sink to the depth of 83.707 inches, or very nearly 7 feet below the surface of the water, before the upward pressure becomes an equipoise for its downward tendency; and this being added to the three feet which it is in length, gives 10 feet for the depth of the body of water, necessary for admitting an equilibrium, under the specified conditions of magnitude and attachment.

CHAPTER X.

OF THE SPECIFIC GRAVITIES OF FLUIDS, AND THE METHOD OF WEIGHING SOLID BODIES BY MEANS OF NON-ELASTIC FLUIDS.

The specific gravity of a body is its weight compared with that of another body of the same magnitude. The magnitude may be expressed by a number denoting its relation to some standard generally used, and as a criterion of comparison, similar to itself, as a cubical inch, a foot, &c.; and if the criteria be different, as in solids and fluids, the magnitudes of bodies are to each other as the criteria multiplied into the numbers expressing these magnitudes. The clear and scientific expositions which in the last chapter were given of absolute and relative weight, must have well prepared the reader for entering upon the doctrine of specific gravities, which distinguishes different species of matter from each other, in one of their most obvious properties, namely, the weight of matter contained in a given space. The weight of any portion of matter is easily ascertained; but it is not always easy to measure the space occupied by a body, or its magnitude; and in some instances this cannot well be effected without artificial means. We employ for this purpose distilled water, the specific gravity of which, or weight of a given bulk, is nearly at all times the same. Adopting, therefore, this pure homogeneous substance as our criterion or unit of measure, by comparing it with other substances the ratio of their specific gravities may be easily discovered; and denoting the specific gravity of water by any number taken at pleasure, the numbers expressing the specific gravities of other bodies are hence given, or, at least, assignable.

PROPOSITION V.

264. When a solid body is immersed in a fluid of different specific gravity from itself:—

> *The weight which the body loses, will be to its whole weight, as the specific gravity of the fluid is to the specific gravity of the solid.*

This is a very important proposition in hydrostatics, but its demonstration does not require the assistance of a diagram; we must therefore endeavour to establish its validity by the application of symbolical arithmetic; for which purpose—

Put m = the magnitude of the immersed solid,
w = its weight,
w'' = the weight lost by the pressure of the fluid,
w' = the weight of a quantity of the fluid, of the same bulk as the solid body,
s = the specific of the solid, and
s' = the specific gravity of the fluid.

Then, because, as we have already stated in a former part of this work, the weights are directly as their magnitudes drawn into their respective gravities; it follows, that

$$w = ms, \text{ and } w' = ms';$$

consequently, by analogy, we obtain

$$w' : w :: ms' : ms.$$

Now, when a body is immersed in a fluid of less specific gravity than itself, it obviously must descend, in consequence of its superior gravity, and the force of descent is equal to the difference between the weight of the solid body and the weight which it loses by the action of the fluid; but it has been shown in the preceding proposition, that the force of descent is equal to the difference between the weight of the solid, and that of an equal bulk of the fluid; consequently, if f denote the force of descent, then in the one case, we have

$$f = w - w',$$

and in the other case, it is

$$f = w - w'';$$

therefore, by comparison, we obtain

$$w - w' = w - w'';$$

hence, by expunging w, it becomes

$$w' = w''.$$

If, therefore, we substitute w'' instead of w', in the first term of the above analogy, we shall get

$$w'' : w :: ms' : ms :: s' : s;$$

hence the truth of the proposition is manifest.

The part of the weight which the body loses by descending in the fluid, is not annihilated, it is only sustained by the upward pressure of the fluid opposing the descent of the body; and this is the reason why the weight of a vessel full of water is not perceptible while it is beneath the surface.

By equating the products of the extreme and mean terms in the preceding analogy, we obtain

$$s w'' = s' w,$$

and dividing by s, we have

$$w'' = \frac{s' w}{s};$$

but according to our notation, w'' denotes the weight which the body loses; consequently, the weight which it retains in the fluid, becomes

$$w - w'' = w - \frac{s' w}{s} = \frac{w(s - s')}{s}. \qquad (184).$$

265. If, therefore, the weight of the body, together with its specific gravity, be known, before it is immersed in a fluid of a given specific gravity; its weight after immersion can easily be ascertained by the following practical rule.

> Rule. *From the specific gravity of the body, subtract that of the fluid in which it is immersed; multiply the remainder by the weight of the body, and divide the product by its specific gravity for the weight which it retains after immersion.*

266. Example. A piece of cast iron which weighs 14 lbs. is plunged into a cistern of water; what force will be required to sustain the iron at rest in any point, its specific gravity being to that of water as 7 to 1?

Here, by operating according to the rule, we have

$$\frac{14(7-1)}{7} = 12 \text{ lbs. avoirdupois.}$$

From which it appears, that 14 lbs. of cast iron being suspended in water, loses 2 lbs. of its weight; or which is the same thing, the upward pressure of the water exceeds its downward pressure, by a force which is equivalent to 2 lbs.

Corol. We may also infer from the above, that the weights which the same body loses by being immersed in different fluids:—

Are as the specific gravities of the fluids.

PROBLEM XXXIV.

267. If a body be weighed in air and in water respectively, and the weights be exactly ascertained:—

> *It is required, from the weights thus exhibited, to determine the real weight.*

Here again, we are under the necessity of dispensing with the assistance of a diagram, the investigation being wholly analytical; in order therefore to proceed,

Put $w =$ the weight of the body when weighed in water,
$w' =$ the weight when weighed in air,
$s =$ the specific gravity of water, generally expressed by unity,
$s' =$ the specific gravity of air, and
$x =$ the real weight of the body required.

Then we have $x - w'$, and $x - w$, for the weights which the body loses in air and in water; but we have deduced it as an inference from Proposition V., that when the same body is weighed in different fluids, it loses weights in proportion to the specific gravities of the fluids in which it is weighed; consequently, we have

$$x - w' : x - w :: s' : s;$$

therefore, by making the product of the mean terms equal to the product of the extremes, we have

$$s\,(x - w') = s'\,(x - w),$$

and from this, by separating the terms, and transposing, we get

$$(s - s')\,x = s\,w' - s'\,w;$$

consequently, by division, we obtain

$$x = \frac{s\,w' - s'\,w}{s - s'}. \qquad (185).$$

268. The equation in its present form, supplies us with the following practical rule for its reduction.

> Rule. *Divide the difference between the products of the alternate weights and specific gravities, by the difference of the specific gravities, and the quotient will be the real weight of the body.*

269. Example. A certain body when weighed in water and in air, is found to equiponderate 12 and 13.9975 lbs. respectively; what is its real weight, the specific gravities of air and water being as 1 to $.001\dot{2}$?

Here, by operating as directed in the rule, we have

$$x = \frac{1 \times 13.9975 - .001\dot{2} \times 12}{1 - .001\dot{2}} = 14 \text{ lbs.}$$

From which it appears, that a body of 14 lbs. avoirdupois, will completely fulfil the conditions of the question.

PROBLEM XXXV.

270. If the weights which a body indicates, when weighed in air and in water, are exactly ascertained:—

> *It is required from thence to determine the specific gravity of the body, the specific gravities of air and water being known.*

Here also, as in the case of the preceding Problem XXXIV., and Proposition V., the aid of a diagram is not required; for it would be totally inconsistent with scientific precision, to denote the specific gravities of bodies by geometrical magnitudes.

Put $\mathrm{W} =$ the real weight of the solid body,
$w =$ the weight when weighed in water,
$w' =$ the weight when weighed in atmospheric air,
$s =$ the specific gravity of water, expressed by unity,
$s' =$ the specific gravity of air, and
$S =$ the required specific gravity of the solid body.

Then, according to the principle announced and demonstrated in the 5th proposition, we have

$$\mathrm{W} - w : \mathrm{W} :: s : S;$$

where it is manifest, that $\mathrm{W} - w$ expresses the weight which the body loses by being weighed in water; therefore, we have

$$\mathrm{W} - (\mathrm{W} - w) : \mathrm{W} :: S - s : S;$$

or by equating the products of the extremes and means, we get

$$Sw = (S - s)\,\mathrm{W};$$

and by proceeding in a similar manner when the body is weighed in air, we obtain

$$Sw' = (S - s')\,\mathrm{W}.$$

Now, from the first of these equations, we have

$$\mathrm{W} = \frac{Sw}{(S - s)},$$

and from the second, it is

$$\mathrm{W} = \frac{Sw'}{(S - s')};$$

hence by compariso

$$\frac{Sw}{(S - s)} =$$

and by taking away the denominators, we get

$$Sw - s'w = Sw' - sw',$$

from which, by transposing and collecting the terms, we obtain

$$(w' - w)S = sw' - s'w,$$

and finally, by division, we have

$$S = \frac{sw' - s'w}{w' - w}. \qquad (186).$$

271. The practical rule or method of reducing this equation, may be expressed in words in the following manner.

> Rule. *Divide the difference between the products of the alternate weights and specific gravities, by the difference of the weights when weighed in air and in water, and the quotient will express the specific gravity of the body.*

272. Example. A certain body when weighed in water indicates exactly 12 lbs. avoirdupois; but when the same body is weighed in air, it indicates 13.9975 lbs.; required the specific gravity of the body, the specific gravities of water and air being as in the preceding problem, or as 1 to .0012?

The process performed according to the directions given in the rule, or after the manner indicated in equation (186), will stand as follows.

$$S = \frac{1 \times 13.9975 - .001\dot{2} \times 12}{13.9975 - 12} = 7.$$

Therefore, a body whose specific gravity is seven times the specific gravity of water, will fulfil the conditions of the question.

PROBLEM XXXVI.

273. If the weights which a solid body indicates, when weighed in air and in water, together with its specific gravity and real weight, are exactly ascertained:—

> *It is required from thence, to determine the magnitude of the body, on the supposition that it is globular.*

If the specific gravity of the body and its real weight were unknown, the solution of the present problem would include that of the two preceding ones; but in order to abbreviate the investigation, we have supposed the specific gravity and the real weight of the body to be given; the process of the solution is therefore as follows.

Put $W =$ the real weight of the globular body,

$S =$ its specific gravity,

$w =$ the weight which the body indicates when weighed in water,

$s =$ the specific gravity of water,

$w' =$ the weight which the body indicates when weighed in air,

$s' =$ the specific gravity of air, and

$d =$ the required diameter of the solid body.

Then, according to the principles of mensuration, the solidity of a globe is expressed by the cube or third power of its diameter, multiplied by the constant decimal .5236; therefore, we have

$$d \times d \times d \times .5236 = .5236 d^3;$$

but it has been stated in a former part of this work, that the absolute weight of any body, is expressed by its magnitude drawn into the specific gravity; hence we have

$$W = .5236 S d^3;$$

consequently, by division, we obtain

$$d^3 = \frac{W}{.5236 S},$$

and from this, by extracting the cube root, we get

$$d = \sqrt[3]{\frac{W}{.5236 S}}. \qquad (187).$$

274. The equation in its present form, expresses the diameter of the body in terms of its absolute weight and specific gravity; this is certainly the simplest and only mode of determining the magnitude of any body or quantity of matter, when the weight and specific gravity are known *à priori;* but when this is not the case, we must have recourse to other methods; and a very elegant and simple one, consists in weighing the body in water and in air, as implied in the problem, and then proceeding as follows.

By equation (185), Problem XXXIV., it appears, that the real or absolute weight of the solid, expressed in terms of its relative weights, and the specific gravities of the fluids in which it is weighed, viz. water and air, is

$$W = \frac{s w' - s' w}{s - s'},$$

and by equation (186), Problem XXXV., the specific gravity of the solid expressed in terms of the same quantities, is

$$S = \frac{sw' - s'w}{w' - w}.$$

But the real or absolute weight of any body, is expressed by its magnitude drawn into the specific gravity; consequently, we have

$$W = \frac{.5236\,d^3\,(sw' - s'w)}{w' - w};$$

let this value of the real weight be compared with that above, and we shall have

$$\frac{.5236\,d^3(sw' - s'w)}{w' - w} = \frac{sw' - s'w}{s - s'}.$$

If the expression $(sw' - s'w)$ be suppressed on both sides of the above equation, we shall obtain

$$\frac{.5236\,d^3}{w' - w} = \frac{1}{s - s'};$$

and again, by suppressing the denominators, we get

$$.5236\,(s - s')\,d^3 = w' - w;$$

therefore, dividing by $.5236\,(s - s')$, we have

$$d^3 = \frac{w' - w}{.5236\,(s - s')},$$

and finally, by extracting the cube root, we obtain

$$d = \sqrt[3]{\frac{w' - w}{.5236\,(s - s')}} \qquad (188).$$

275. Now, the methods of reducing the equations (187) and (188), or the practical rules derived from them, may be expressed as follows.

1. When the absolute weight and specific gravity are given.

   RULE. *Divide the absolute weight of the body, by* .5236 *times the specific gravity, and the cube root of the quotient will be the diameter of the solid sought.*

2. When the weights indicated by the body in water and in air are given.

   RULE. *Divide the difference between the weights, as obtained from weighing the body in air and in water, by* .5236 *times the difference between the specific gravities of water and air; then, the cube root of the quotient will be the diameter of the solid sought.*

276. EXAMPLE 1. The absolute weight of a globular body is 14 lbs., and its specific gravity 7; what is its diameter?

This example corresponds to equation (187), and must therefore be resolved by the first case of the foregoing rule, observing to bring the numerator into the same denomination with the denominator, that is, reducing lbs. avoirdupois into ounces; or thus, $14\times16=W$, the absolute weight, from which we get

$$d=\sqrt[3]{\frac{(14\times16)}{.5236\times7}}=3.9313,\text{ or nearly 4 inches.}$$

Hence it appears, that a globular body, whose specific gravity is seven times greater than that of water, will weigh 14 lbs. when its diameter is 3.9313 inches, which corresponds very nearly with a globe of cast iron.

277. Example 2. A globular body whose specific gravity and absolute weight are unknown, indicates 12 lbs. avoirdupois when weighed in water, and 13.9975 lbs. when weighed in air; what is its magnitude, the specific gravity of water and air being to one another as the numbers 1 and $.001\dot{2}$?

This second example corresponds to the conditions represented in equation (188), and must therefore be resolved by the second case of the foregoing rule, the numerator being brought into the same denomination with the denominator, or the lbs. avoirdupois being turned into ounces, as $(13.9975-12)\,16$, from which we obtain

$$d=\sqrt[3]{\frac{(13.9975-12)\,16}{.5236(1-.001\dot{2})}}=3.9313\text{ or nearly 4 inches, the same}$$

as above.

From the principles established in the foregoing Proposition (V), and the problems derived from it, we deduce the following inferences.

278. Inf. 1. When bodies of equal weights, but of different magnitudes, are immersed in the same fluid :—

> *The weights which they lose, are reciprocally proportional to their specific gravities, or directly proportional to their magnitudes.*

279. Inf. 2. When a solid body is weighed in air, or in any other fluid whatever :

> *The difference between its absolute weight, and the weight exhibited in the fluid, is the same as the weight of an equal bulk of the fluid.*

280. Inf. 3. If two solid bodies of different magnitudes, when weighed in the same fluid indicate equal weights :—

> *The greater body will preponderate when they are brought into a rarer medium.*

281. INF. 4. If two solid bodies of different magnitudes, indicate equal weights when weighed in the same fluid :—

> *The lesser body will preponderate when they are placed in a denser medium.*

282. INF. 5. If two or more solid bodies, when placed in the same fluid, sustain equal diminutions of weight :—

> *The magnitudes of the several bodies are equal among themselves.*

This is manifest, for the losses of weights are as the weights of the quantities of fluid displaced; and these are as the magnitudes of the bodies which displace them.

PROBLEM XXXVII.

283. If two bodies of equal weights, but different specific gravities, be exactly equipoised in air, and then immersed in a fluid of greater specific gravity, the smaller body will prevail :—

> *It is therefore required to determine, what weight must be added on the part of the greater body, to restore the equilibrium.*

Put s = the specific gravity of the fluid, in which the bodies are immersed, after being equipoised in air,
s' = the specific gravity of the greater body,
s'' = the specific gravity of the smaller body,
w = the common weight of each,
w' = the weight lost by the greater body, by reason of the immersion,
w'' = the weight lost by the lesser body, and
x = the weight which must be added to restore the equilibrium.

Then, because by the preceding proposition, when a body is immersed in a fluid, the weight which it loses, is to its whole weight, as the specific gravity of the fluid is to that of the body; it follows, that

$$s' : s :: w : w';$$

and this, by reducing the proportion, gives $w' = \frac{ws}{s'}$,

Again, for the weight lost by the lesser body, we have

$$s'' : s :: w : w'';$$

which by reduction gives

$$w'' = \frac{ws}{s''}.$$

Now, it is manifest, that the weight required to restore the equilibrium, must be equal to the difference between the results of the above analogies; therefore, we obtain

$$w' - w'' = x = \frac{ws}{s'} - \frac{ws}{s''};$$

which, by a little farther reduction, becomes

$$x = \frac{ws(s'' - s')}{s' s''}. \qquad (189).$$

284. The practical rule which this equation affords, may be expressed in words at length in the following manner.

> RULE. *Multiply the difference between the specific gravities of the bodies, by their common weight in air, drawn into the specific gravity of the fluid in which they are immersed, and divide the result by the product of the specific gravities of the bodies, for the weight to be added in order to restore the equilibrium.*

It may be proper here to observe, that the weight determined by this rule must not be immersed in the fluid, it must only be attached to that side of the balance on which the greatest weight is lost.

285. EXAMPLE. Suppose that 84 lbs. of brass, whose specific gravity is 8.1 times greater than that of water, is equipoised in air by a piece of copper, whose specific gravity is 9 times greater than that of water; how much weight must be applied to the ascending arm of the balance to restore the equilibrium, the same being destroyed by immersing the bodies in water, of which the specific gravity is expressed by unity?

Here, by attending to the directions in the rule, we get

$$x = \frac{84 \times 1 \times (9 - 8.1)}{8.1 \times 9} = 1.0\dot{3}\dot{7} \text{ lbs.}$$

Hence it appears, that if a mass of brass and of copper, each equal to 84 lbs. when weighed in air, be immersed in a vessel of water, the copper will preponderate, in consequence of its greater specific gravity; and in order that the equilibrium may be again restored, a weight of $1.0\dot{3}\dot{7}$ lbs. must be attached to the ascending arm of the balance, or that from which the brass is suspended.

PROBLEM XXXVIII.

286. If two bodies of different, but known specific gravities, equiponderate in a fluid of given density:—

It is required to determine the ratio of the quantities of matter which they contain.

Put $s =$ the specific gravity of the fluid, in which the bodies are found to equiponderate,
$m =$ the magnitude of the greater body,
$s' =$ its specific gravity,
$m' =$ the magnitude of the lesser body,
$s'' =$ its specific gravity,
$w =$ the weight of the greater body in the fluid, and
$w' =$ the weight of the lesser body under the same circumstances.

Then, by Proposition V., when a solid body is immersed in a fluid of different specific gravity, the weight which it loses, is to its whole weight, as the specific gravity of the fluid, is to the specific gravity of the solid; it therefore follows, that

$$s' : s :: ms' : ms = \text{the weight lost by the greater body};$$

but the weight of the body in the fluid, is manifestly equal to the difference between its absolute weight, and that which it loses in consequence of the immersion; hence we have

$$w = ms' - ms = m(s' - s);$$

and by a similar mode of procedure, we obtain

$$s'' : s :: m's'' : m's = \text{the weight lost by the lesser body};$$

consequently, the weight which it possesses in the fluid, is

$$w' = m's'' - m's = m'(s'' - s).$$

Now, according to the conditions of the problem, these are in equilibrio with one another; therefore by comparison, we have

$$m(s' - s) = m'(s'' - s),$$

and by converting this equation into an analogy, it is

$$m : m' :: (s'' - s) : (s' - s);$$

and finally, if we multiply the first and third terms by s', and the second and fourth by s'', we shall have

$$ms' : m's'' :: s'(s'' - s) : s''(s' - s).$$

287. Example. Twenty ounces of brass, whose specific gravity is eight times greater than that of water, and a piece of copper whose

specific gravity is nine times greater, are in equilibrio with one another in a fluid whose specific gravity is unity; required the weight of the copper?

Here we have given $m s'=20$ ounces; $s'=8$; $s''=9$, and $s=1$; consequently, by substitution, the above analogy becomes

$$20 : m's'' :: 8(9-1) : 9(8-1);$$

and by equating the products of the extremes and means, we get

$$64m's''=1260,$$

and dividing by 64, we have

$$m's''=\frac{1260}{64}=19\tfrac{11}{16} \text{ ounces.}$$

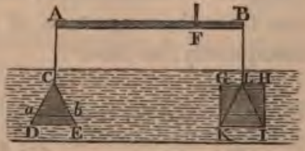
It therefore appears, that 20 ounces of brass and $19\frac{11}{16}$ ounces of copper, are in equilibrio with each other, when immersed in a fluid whose specific gravity is unity; but if put into a fluid of greater density, the copper will prevail.

PROBLEM XXXIX.

288. Suppose a cylinder and cone, of the same altitude, base, and specific gravity, to balance each other at the extremities of a straight lever, when immersed in a fluid of given density; the cone being suspended at the vertex, and the cylinder at the extremity of the axis. Now, suppose a cone equal to the one proposed, to be abstracted from the cylinder, and its place supplied by another of the same magnitude and half the specific gravity; it is manifest that in this state, the cone will preponderate:—

> *It is therefore required to determine, how much must be taken from the cone, in order that the equilibrium may be again restored.*

Let AB be a straight inflexible lever, supported upon and easily moveable about the fulcrum F, and let the cone CDE and the cylinder GHIK, (equal in altitude, base, and specific gravity,) be suspended from the extremities at A and B.



Then it is manifest, that in consequence of the equality of the bases and altitudes, the magnitude of the cylinder is equal to three times the magnitude of the cone; and since the specific gravities of the bodies are the same, it follows also, that the weight of the cylinder is equal to three times the weight of the cone; consequently, by the principles of the lever, the length of the arm AF, is three times the length of the arm BF; for it is a well known property in the doctrine of mechanics, that when two bodies of different weights are in equilibrio on the opposite arms of a straight lever:—

> *The lengths of the arms are to each other, reciprocally as the weights of the suspended bodies.*

Now, suppose the cone LKI, which is obviously equal in magnitude to CDE, to be abstracted from the cylinder, and to have its place supplied by another cone of half the specific gravity as the former; then it is evident, that if the cone CDE is suffered to retain its magnitude, it will preponderate and cause the cylinder to ascend; it is therefore necessary, in order that the equilibrium shall not be disturbed, to diminish the magnitude, and consequently the weight of the equilibrating cone; and for the purpose of assigning the quantity of diminution,

Put $m =$ the magnitude of the conical body CDE,
$m' =$ the magnitude of the cylindrical body GHIK,
$m'' =$ the magnitude of the remaining portion Cab,
$w =$ the weight which the cone loses in the fluid,
$w' =$ the weight lost by the cylinder,
$w'' =$ the weight lost by the remaining cone Cab,
$s =$ the specific gravity of the fluid, and
$s' =$ the specific gravity of the cone and cylinder.

Then, since the weight which a body loses by being immersed in a fluid, is to its whole weight, as the specific gravity of the fluid is to the specific gravity of the body, we have

$$s' : s :: ms' : w;$$

therefore, by equating the products of the extremes and means, it is

$w = ms =$ the weight lost by the cone; but according to the principles of mensuration, the magnitude of a cylinder is equal to three times the magnitude of a cone of the same base and altitude; consequently, we have

$$m' = 3m,$$

and for the weight lost by the cylinder, we get

$$s' : s :: 3ms' : w';$$

from which, by equating the product of the extremes and means, we obtain

$$w' = 3ms = m's,$$

and in like manner, the weight lost by the cone $\text{c}ab$, is found to be

$$w'' = m''s.$$

But the weights which the several bodies possess in the fluid, are manifestly equal to the difference between the absolute weights and the weights lost; and the absolute weights are equal to the magnitudes drawn into the specific gravities; therefore, we have

$ms' - ms = m(s' - s) =$ the weight of the cone in the fluid,
$3ms' - 3ms = 3m(s' - s) =$ the weight of the cylinder,
$m''s' - m''s = m''(s' - s) =$ the weight of the remaining cone.

Now, if from the weight which the cylinder possesses in the fluid, we subtract the corresponding weight of the cone, and to the remainder add the weight of another cone of equal magnitude and half the specific gravity; then, the reduced weight of the cylinder in the fluid becomes

$$2ms' + \tfrac{1}{2}ms' - 3ms = m(2\tfrac{1}{2}s' - 3s).$$

But according to the conditions of the problem, this weight is to be in equilibrio with the weight of the remaining cone; therefore, by the property of the lever, we have

$$m(2\tfrac{1}{2}s' - 3s) : m''(s' - s) :: 3 : 1;$$

and from this, by equating the products of the extremes and means, we get

$$3m''(s' - s) = m(2\tfrac{1}{2}s' - 3s),$$

in which equation m'' is unknown; in order therefore to determine its value, divide both sides of the equation by $3(s' - s)$, and it becomes

$$m'' = \frac{m(5s' - 6s)}{6(s' - s)}. \qquad (190).$$

But this that we have determined, is the magnitude of the part which remains, whereas the problem requires the magnitude of the part to be cut off; now, the magnitude of the whole cone is m; consequently, by subtraction, we have

$$m - m'' = m - \frac{m(5s' - 6s)}{6(s' - s)} = \frac{ms'}{6(s' - s)}. \qquad (191).$$

289. The practical rule for reducing this equation is very simple, it may be expressed in the following manner.

> RULE. *Multiply the magnitude of the cone by its specific gravity, and divide the product by six times the difference between the specific gravity of the cone and cylinder, and that of the fluid, and the quotient will give the magnitude of the part to be cut off, in order to restore the equilibrium.*

290. EXAMPLE. Suppose that a cone and cylinder of copper, whose specific gravity is nine times greater than that of water, are immersed in a fluid, whose specific gravity is 1.85, and placed under the conditions specified in the problem; how much must be cut from the lower part of the cone to restore the equilibrium, the diameter of the bases and the altitude of the cone and cylinder being respectively 2 and 5 inches?

Here, by operating according to the rule, we shall have

$$m - m'' = \frac{5.236^{*} \times 9}{6(9-1.85)} = \frac{47.124}{42.9} = 1.098 \text{ cubic inches,}$$

the part to be cut off from the cone, in order that the remainder may equipoise the cylinder; or we may calculate the magnitude of the equipoising cone by equation (190), in the following manner.

$$m'' = \frac{5.236(5 \times 9 - 6 \times 1.85)}{6(9 - 1.85)} = 4.135 \text{ cubic inches;}$$

which, by subtraction, gives 1.098 cubic inches for the part to be cut off.

Put $d =$ the diameter of the base of the cone and cylinder, and
$h =$ the common height or altitude.

Then, the equations (190) and (191) will become transformed, in terms of the dimensions, into those that follow, viz.

$$m'' = \frac{.2618 d^2 h(5s' - 6s)}{6(s' - s)}. \qquad (192).$$

This equation expresses the magnitude of the cone which restores the equilibrium, and the following one expresses the magnitude of the frustum which has to be deducted, in order that the equilibrium may obtain; that is,

$$m - m'' = \frac{.2618 d^2 h s'}{6(s' - s)}. \qquad (193).$$

The above is a better mode of expressing the quantities, than that exhibited in equations (190) and (191); since it is not probable, that the magnitudes or solid contents of the bodies will be proposed, without having previously stated their linear dimensions.

It would be superfluous to propose an example, for the purpose of illustrating the reduction of the equations in their modified form; for since the expression $.2618 d^2 h$, is equivalent to the magnitude of the cone, the rule in words would be the same in both cases, and therefore, it need not be repeated.

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* The number 5.236 is that which expresses the magnitude of the cone, for $2^2 \times .7854 \times 5 \times \frac{1}{3} = 5.236$.

PROBLEM XL.

291. If a solid body be weighed in vacuo and in a fluid, and the different weights correctly noted :—

> *It is required from thence, to compare the specific gravities of the solid, and the fluid in which it is immersed.*

The solution of this problem is extremely easy, for the difference between the weight of the body in vacuo and in the fluid, gives the weight lost; therefore,

Put $w =$ the weight indicated by the body when weighed in vacuo,
$w' =$ the weight when weighed in the fluid,
$s =$ the specific gravity of the fluid in which the body is weighed, and
$s' =$ the specific gravity of the body.

Then we have $w - w' =$ the weight which the body loses by being weighed in the fluid; therefore, by the fifth proposition, we obtain

$$w - w' : w :: s : s'; \text{ that is}$$

$$\frac{w - w'}{w} = \frac{s}{s'}; \qquad (194).$$

consequently, since the one ratio is given, the latter can be found.

292. Example. Suppose a piece of metal to indicate 40 ounces when weighed in vacuo, and 35 ounces when weighed in water; what is the specific gravity of the metal?

Here, by substituting the given numbers in equation (194), we get

$$\frac{40 - 35}{40} = \frac{1}{8} = \frac{s}{s'};$$

hence, the specific gravity of the solid, is eight times greater than the specific gravity of water.

PROBLEM XLI.

293. If two solid bodies be weighed in vacuo and in a fluid, and the different weights correctly noted :—

> *It is required from thence, to compare the specific gravities of the bodies.*

The intelligent reader will readily perceive, that the present problem is only an extension of that which immediately precedes it, and is proposed with the design of detecting the law, by which the specific gravities of different bodies are compared; for which purpose,

Put $W =$ the weight of the heavier body when weighed in vacuo,
$W' =$ its weight when weighed in the fluid,
$w =$ the weight of the lighter body when weighed in vacuo,
$w' =$ its weight when weighed in the fluid,
$s =$ the specific gravity of the fluid in which the bodies are weighed,
$s' =$ the specific gravity of the heavier body, and
$s'' =$ the specific gravity of the lighter.

Then, the weights which the bodies lose by being weighed in the fluid, are respectively $W - W'$, and $w - w'$, and because the weight lost, is to the whole weight, as the specific gravity of the fluid is to the specific gravity of the solid; it follows in the case of the heavier body, that

$$W - W' : W :: s : s',$$

and in the case of the lighter body, it is

$$w - w' : w :: s : s'';$$

therefore, by equating the products of the extreme and mean terms in each of these analogies, we have

$$s'(W - W') = sW, \text{ and } s''(w - w') = sw;$$

consequently, by division, we obtain

$$s' = \frac{sW}{W - W'}, \text{ and } s'' = \frac{sw}{w - w'};$$

hence, by analogy, we have

$$s' : \frac{sW}{W - W'} :: s'' : \frac{sw}{w - w'},$$

and finally, by suppressing s, it becomes

$$s' : \frac{W}{W - W'} :: s'' : \frac{w}{w - w'}.$$

Hence it appears, that the specific gravities of the two bodies, are to one another, as their absolute weights divided by the weights which they lose in the fluid; and it is manifest, that the same law will extend to any number of bodies whatever; therefore, the method of comparison has been determined.

294. **Example.** A solid body whose absolute weight is 23 ounces, when weighed in a certain fluid, loses 3 ounces of its weight; and another body of 800 ounces, when weighed in the same fluid, loses 102 ounces; what is the ratio of their respective gravities?

Here, since the loss of the one is 3 ounces, and that of the other 102 ounces, it follows, that the specific gravities of the bodies, are to one another as the numbers 391 and 400; for we have

$$\frac{23}{3} : \frac{800}{102} :: 391 : 400.$$

And in like manner, if three or more bodies be weighed in the same fluid, their specific gravities may be compared with one another, and also with that of the fluid in which they are weighed.

PROBLEM XLII.

295. If a solid body of known specific gravity, be weighed in several different fluids, and the weights correctly indicated:—

It is required from thence, to determine the ratio of their respective gravities.

This problem, it will readily be perceived, is exactly the reverse of the preceding one, and therefore, the method of its solution may easily be discovered; it is, however, of equal utility in philosophical inquiries, for which reason we have proposed it in this place.

Put $W =$ the weight of the solid when weighed in vacuo,
$s =$ the specific gravity of the solid,
$w =$ the weight which it indicates in a fluid whose specific gravity is s', and
$w' =$ the weight which it indicates in a fluid whose specific gravity is s''.

Then the weights which the body loses, by being weighed in the two fluids, are respectively $W - w$ and $W - w'$; but the weight lost, is to the whole weight, as the specific gravity of the fluid is to the specific gravity of the solid; hence, for the first fluid, we have

$$W - w : W :: s' : s;$$

from which, by equating the products of the extremes and means, we get

$$s'W = s(W - w);$$

therefore, by division, we obtain

$$s' = \frac{s(W - w)}{W}. \qquad (195).$$

Again, for the second fluid, we have

$$W - w' : W :: s'' : s,$$

and from this, by equating the products of the extremes and means, we get

$$s'' W = s(W - w'),$$

and this by division becomes

$$s'' = \frac{s(W - w')}{W}; \qquad (196).$$

hence, by analogy, we obtain

$$s' : \frac{s(W - w)}{W} :: s'' : \frac{s(W - w')}{W};$$

and finally, by suppressing the common quantities in the second and fourth terms, we get

$$s' : W - w :: s'' : W - w';$$

that is, the specific gravities of different fluids, are as the weights which the body loses.

296. EXAMPLE. A mass of brick whose absolute weight is 64 ounces, and its specific gravity equal to twice the specific gravity of water; when weighed in one fluid indicates 37 ounces, and when weighed in another, it indicates only 30 ounces; it is required from thence, to determine the ratio of the respective gravities of the fluids, and also the specific gravity of each?

Here it is manifest, that the weight lost by the solid when weighed in one fluid, is

$$W - w = 64 - 37 = 27 \text{ ounces},$$

and on being weighed in another fluid, it loses

$$W - w' = 64 - 30 = 34 \text{ ounces}.$$

Now, we have seen above, that the specific gravities of the fluid in which the solid is weighed, are to one another, respectively as the weights which the solid loses in them; consequently, we have

$$s' : s'' :: 27 : 34.$$

This is the ratio of the specific gravities; but it appears from equations (195) and (196), that when the specific gravity of the solid is known, the specific gravity of the fluid in which it is weighed can easily be ascertained.

If, therefore, we employ the specific gravity of the body as given in the question, the specific gravity of the first fluid, by equation (195), becomes

$$s' = \frac{2(64-37)}{64} = .844 \text{ nearly.}$$

and for the specific gravity of the second fluid, it is

$$s'' = \frac{2(64-30)}{64} = 1.062 \text{ nearly.}$$

Hence, the absolute specific gravities of the fluids, are respectively equal to 0.844 and 1.062, that of water being unity; and if we refer to a table of the specific gravities of known fluids, we will find these numbers to correspond with alcohol and acetic acid, the two fluids in which the brick is weighed.

It is manifest, that what has been done above in respect of two fluids, may be extended to any number of fluids whatever, the law, and the method of determining the specific gravity, being the same in all.

PROBLEM XLIII.

297. If a body be weighed in air, and again in a vessel filled with water, the weight of the vessel and water being known:—

It is required from thence, to determine the specific gravity of the body.

The principle upon which the solution of this problem depends is not so evident; it may, nevertheless, be briefly expounded in the following manner.

The body is first weighed in air; then being put into a vessel filled with water, the weight of which is known, it will expel a quantity of the fluid equal to its own bulk, and because the specific gravity of the body is supposed to be greater than that of water, it is obvious, that the vessel and its contents will now be heavier than it was before the body was put into it, by a quantity, which is equal to the difference between the weight of the body, and that of an equal bulk of the water; but the body loses as much of its own weight in the fluid, as is equal to that of the water displaced; hence, the determination of its specific gravity becomes easy, for which purpose,

Put w = the weight of the body when weighed in air,
w' = the weight of the vessel and the water before the body is put into it,
w'' = the weight of the vessel, the water, and the solid, and
s = the specific gravity of the solid.

Then the weight gained by the vessel, by reason of the immersion of the solid body, is $w'' - w'$, and this expresses the weight of the body in the fluid; consequently, the weight which the body loses, is $w - (w'' - w')$.

But by the 5th Proposition preceding, the weight lost, is to the whole weight, as the specific gravity of the fluid is to the specific gravity of the body; therefore, because the specific gravity of water is expressed by unity, we have

$$w - (w'' - w') : w :: 1 : s; \text{ that is}$$

$$s = \frac{w}{w - (w'' - w')}. \qquad (197).$$

298. The practical rule for reducing the above equation, may be expressed in words at length in the following manner.

> Rule. *Divide the weight of the body in air, by the difference between that weight, and what is gained by the vessel in consequence of the immersion, and the quotient will express the specific gravity of the solid.*

299. Example. A solid body when weighed in air, indicates a weight of 16 ounces; and when put into a vessel filled with water, the vessel, the solid and the water together, indicate a weight of 36 ounces; whereas the vessel when filled with water alone weighs only 32 ounces; required the specific gravity of the body, that of water being expressed by unity?

Here, by following the directions of the rule, we have

$$s = \frac{16}{16 - (36 - 32)} = 1\tfrac{1}{3},$$

being a measure of specific gravity, which corresponds very nearly with American ebony, a very suitable material for hydrostatical experiments.

300. We have hitherto been considering the nature of bodies that are specifically heavier than the fluids in which they are weighed, and consequently, such as would sink to the bottom, if they were left to the free action of their own gravity; we have therefore, in the next place, to consider such bodies as are specifically lighter than the fluids on which they are placed, and consequently, such as would float on the surface, if left to the free exercise of their own buoyancy.

This is a very abstruse, but interesting and important department of Hydro-Dynamical science; for on it depends the principles by which we determine the conditions of equilibrium, and the stability of floating bodies.

PROBLEM XLIV.

301. If a solid body is weighed in a fluid of greater specific gravity than itself, and of which the specific gravity is given:—

It is required from thence, to determine the specific gravity of the solid.

The usual method of resolving this problem, is, to attach the body whose specific gravity is required, to another body specifically heavier than the fluid, and of a sufficient magnitude to cause the compound mass to sink; then, by observing the weights indicated by the subsidiary body, and by the compound mass, when they are separately placed in the fluid, the specific gravity of the lighter body will become known.

Put $w =$ the weight of the lighter body when weighed in vacuo,
$w' =$ the difference between the weight of the compound mass, and that of the heavier body, when weighed separately in the fluid,
$s =$ the specific gravity of the fluid, and
$s' =$ the specific gravity of the solid required.

Now, it is manifest, that the effort of buoyancy, or the force of ascent of the lighter body, is equal to the difference between the weight of the compound mass in the fluid, and that of the heavier body in the fluid; therefore,

$w' =$ the force of ascent of the lighter body.

But, the force of ascent of the lighter body, or, as it is more elegantly denominated, the effort of buoyancy, is evidently equivalent to the excess of the weight of a quantity of the fluid, equal in magnitude to the lighter body, above the weight of the lighter body when weighed in vacuo; consequently, the weight of a quantity of the fluid, equal in bulk to the lighter body, is expressed by $w + w'$; hence, we have

$$w + w' : w :: s : s',$$

from which, by reducing the proportion, we get

$$s' = \frac{sw}{w + w'}.$$

Or put $W =$ the weight of the compound mass when weighed in the fluid, and $W' =$ the corresponding weight of the heavier body when weighed separately; then we have

$$w' = W - W';$$

let this value of the force of ascent, be substituted instead of it, in the above value of s', and we shall obtain

$$s' = \frac{s w}{w + W - W'}. \qquad (198).$$

302. The practical rule for reducing this equation, may be expressed in words at length in the following manner.

> RULE. *Multiply the absolute weight of the body, of which the specific gravity is required, by the specific gravity of the fluid; then, divide the product by the absolute weight of the body, increased by its force of ascent, and the quotient will be the specific gravity sought.*

303. EXAMPLE. A piece of wood which weighs in vacuo 22 ounces, is attached to a piece of metal of such a magnitude as to weigh 12 ounces in water; now, supposing that when the compound mass is placed in water, it is found to weigh 20 ounces; what is the specific gravity of the wood, that of water being expressed by unity?

Here, by proceeding according to the rule, we get

$$s' = \frac{1 \times 22}{22 + 20 - 12} = 0.733;$$

and this, by referring to a table of specific gravity, is found to correspond very nearly with the medium species of citron wood.

PROBLEM XLV.

304. Having given the weight of a vessel full of water, both before and after a body of a given weight in air is immersed in it, together with the specific gravity of the air at the time of observation:—

> *It is required to determine the specific gravity of the immersed solid, the weight of the air being taken into consideration.*

This problem is perhaps more curious than useful; but since it tends to excite the reader's attention, and to render him familiar with the resources of analysis, we have thought proper to introduce it in this place; and in order to its resolution,

Put $w =$ the weight of the vessel when full of water,
$w' =$ the weight of the solid body when weighed in air,
$w'' =$ the weight of the vessel with the solid in it, when filled up with water,

Put $m =$ the magnitude of the solid,

$s =$ its specific gravity, or the quantity which the problem demands, and

$s' =$ the specific gravity of air at the time of the experiment.

Then, because the weight of any body, is expressed by the product of its magnitude drawn into its specific gravity; it follows, that when the weight of the air is disregarded, the weight of the solid is

$$w' = ms;$$

but the weight of a quantity of air equal in magnitude to the body, is ms', and it evidently loses as much weight as that of the fluid which it displaces; consequently, when the weight of the air is considered, the weight of the body when weighed in air, becomes

$$w' = m(s - s'),$$

therefore, by division, the magnitude of the solid is

$$m = \frac{w'}{s - s'};$$

consequently, the weight in vacuo, is

$$ms = \frac{sw'}{s - s'}.$$

Then, because w'' denotes the weight of the vessel with the solid in it, when filled up with water, and w the weight of the vessel when full of water; then $w'' - w$ expresses the weight which the vessel has gained by the immersion of the solid, and this is manifestly equal to the difference between the weight of the solid, and that of an equal bulk of the fluid; therefore, the weight of a quantity of water, equal in bulk to the solid, is

$$\frac{sw'}{s - s'} - w'' + w = \frac{s(w + w' - w'') + s'(w'' - w)}{s - s'}.$$

Then, as the specific gravity of the body, is to the specific gravity of water, so is the weight of the body, to the weight lost; that is,

$$s : 1 :: \frac{sw'}{s - s'} : \frac{s(w + w' - w'') + s'(w'' - w)}{s - s'};$$

and this, by suppressing the denominator in the homologous terms, becomes

$$s : 1 :: sw' : s(w + w' - w'') + s'(w'' - w),$$

and by equating the products of the extreme and mean terms, we obtain

$$s(w + w' - w'') + s'(w'' - w) = w';$$

therefore, by transposition, we get

$$s(w + w' - w'') = w' - s'(w'' - w),$$

and finally, by division, we shall obtain

$$s = \frac{w' - s'(w'' - w)}{w + w' - w''}. \qquad (199).$$

305. The practical rule by which the reduction of the above equation is effected, may be expressed in words at length in the following manner.

> RULE. *From the weight of the vessel with the solid in it, when filled up with water, subtract the weight of the vessel when full of water only; then multiply the remainder by the specific gravity of the air at the time of observation, and subtract the product from the weight of the solid in air for a first number.*
>
> *To the weight of the vessel when full of water, add the weight of the solid when weighed in air, and from the sum, subtract the weight of the vessel with the solid in it, when filled up with water, and the remainder will be a second number.*
>
> *Divide the first number by the second, and the quotient will give the specific gravity of the solid.*

306. EXAMPLE. The weight of a vessel when full of water is 68 lbs. avoirdupois, and the weight of a solid body when weighed in air of a medium temperature, is 34 lbs.; now, when the solid is placed in the vessel, its bulk of water is expelled, and the vessel being then weighed, is found to indicate 86 lbs.; required the specific gravity of the solid body?

When the air is of a medium temperature, its specific gravity is very nearly expressed by the fraction $0.001\dot{2}$, that of water being unity; therefore, by proceeding according to the rule, we have

$$s = \frac{34 - .001\dot{2}(86 - 68)}{68 + 34 - 86} = 2.123,$$

which, by referring to a table of specific gravities, is found to correspond very nearly with the opal stone, a silicious material of very great value, for the senator Nonius preferred banishment to parting with his favourite *opal*, which was coveted by Antony.

CHAPTER XI.

OF THE EQUILIBRIUM OF FLOATATION.

By the *equilibrium of floatation* is generally meant the position of a floating body, when its centre of gravity is in the same vertical line with the centre of gravity of the displaced fluid. When the lower surface of the floating body is spherical or cylindrical, the centre will coincide with the centre of the figure; as, in all circumstances, the height of this point, as well as the form of the volume of fluid displaced, must remain invariable. In the next proposition, we shall prove, that the place of the centre is determined by the doctrine of forces combined with the elementary principles of hydrostatics, by considering the form and extent of the surface of the displaced portion of the fluid, compared with its bulk, and with the situation of its centre of gravity. Our inquiry will be found to embrace also rectangular figures; solids in the form of paraboloids and cylinders, together with the equilibrium of floatation of solids immersed in fluids of different specific gravities; the theory of construction of the *aërometer;* the *hydrostatic balance*, and the method of weighing solid bodies in fluids. The reader will therefore consider this syllabus as the paraphrase of a definition for the term *equilibrium of floatation.*

PROPOSITION VI.

307. When a body floats in a state of equilibrium on the surface of an incompressible and non-elastic fluid:—

> *The centre of gravity of the whole body, and that of the part immersed; or which is the same thing, that of the fluid displaced, must occur in the same vertical line.*[*]

Let ADBH be a vertical section passing through G and g, the centres of gravity of the whole body AHBD, and of the immersed part ADB, which falls below EF the plane of floatation.

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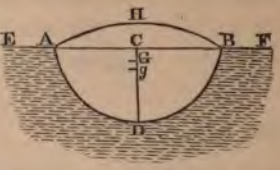
[*] The vertical line, which passes through the centres of gravity of the whole body, and the part immersed beneath the plane of floatation, is generally denominated the *line of support.*



For since g is the centre of gravity of the immersed part of the body, or of the fluid ADB, it is also the centre of all the forces or weights of the particles of fluid in ADB, tending downwards; but because the body is at rest, the same point g is also the centre of all the pressures of the fluid tending upwards, by which the weight of the body ADBH is sustained in a state of equilibrium.

Now, it is manifest, that the sum of all the downward forces, is equal and opposite to the sum of all the upward forces, otherwise the body could not be in a state of rest; but the direction in which the weight of the body tends downwards from C, is perpendicular to the horizon; consequently, the line CD which passes through G and g, the centres of gravity of the whole body and of the part immersed, must also be perpendicular to the horizon; for if it is not, the body must have a rotatory motion, but according to the hypothesis, the body is at rest; therefore, the line CD is perpendicular to the horizon.

From Propositions III. and VI., it is obvious, that for a floating body to remain at rest, or in a state of equilibrium, two conditions must obtain, and these are,

> *The weight of the floating body, and that of the fluid displaced, must be equal to one another.*

This is manifest from the inference under the third Proposition, and the second condition to which we have alluded forms the substance of Proposition VI., viz.

> *The centre of gravity of the whole body and that of the part immersed, or of the fluid displaced, occur in the same vertical line.*

It is extremely obvious, from the nature of the subject, that both the above conditions must have place; for if the first do not obtain, the body will ascend or descend in a direction which is perpendicular to the horizon; and if the second fail, the body will turn about its centre of gravity, until the centre and that of the fluid displaced occur in the same vertical line; and if both these conditions fail together, the body will partake of a progressive and a rotatory motion at one and the same time.

From the Proposition which we have demonstrated above, two or three very useful inferences may be deduced, as follows.

308. Inf. 1. If any homogeneous plane figure be divided symmetrically by its vertical axis, and placed in a fluid of greater specific gravity than itself:—

It will remain in equilibrio with its bisecting axis vertical.

309. Inf. 2. If any homogeneous solid, generated by the revolution of a curve round its vertical axis, be placed in a fluid of greater specific gravity than itself:—

It will remain in equilibrio in that position, that is, with its axis vertical.

310. Inf. 3. If in any homogeneous prismatic body, whose axis is horizontal, the centre of gravity of the section made through its middle parallel to its base, be in the same vertical line with the centre of gravity of that part of the solid which falls below the plane of floatation:—

The body will remain in equilibrio in that position, if placed in a fluid of greater specific gravity than itself.

This is manifest, for the centres of gravity of the whole prism, and of the part immersed, may be conceived to lie in those points, and consequently, the prismatic body is in a state of equilibrium.

PROPOSITION VII.

311. When a solid body floats upon a fluid of greater specific gravity than itself, and has attained a state of equilibrium:—

The magnitude of the body, is to that of the part immersed below the plane of floatation, as the specific gravity of the fluid is to that of the floating body.

For by the inference to the third proposition, when the body floats in a state of equilibrium:—

The weight of the floating body, is equal to the weight of a quantity of the fluid, whose magnitude is the same as that portion of the solid which falls below the plane of floatation.

And according to this principle, the truth of the above Proposition is demonstrated; for,

Put $m =$ the magnitude of the whole floating body,
$m' =$ the magnitude of the part immersed,
$s =$ the specific gravity of the floating body,
$w =$ its weight,

s' = the specific gravity of the fluid, and

w' = the weight of a quantity of the fluid, of the same magnitude as that part of the body which falls below the plane of floatation; then, according to the above inference just stated, we get

$$w = w'.$$

But because the weight of any body is expressed by the product of its magnitude drawn into its specific gravity; it follows, that

$$w = ms, \text{ and } w' = m's',$$

consequently, by comparison, we have

$$ms = m's'. \qquad (200).$$

Therefore, if this equation be converted into an analogy, the truth of the Proposition will become manifest; for

$$m : m' :: s' : s,$$

being precisely the principle which the Proposition implies.

From the principle demonstrated above, various curious and interesting questions may be resolved, and by selecting a few which point directly to practical subjects, the information afforded by their resolution will sufficiently repay the labour of an attentive perusal.

312. EXAMPLE 1. A cubical block of fir, whose specific gravity is 0.55, floats in equilibrio on the surface of a fluid whose specific gravity is 1.026; how much of the block is above, and how much below the plane of floatation, the entire magnitude being equal to 512 cubic inches?

Here, by the Proposition, we have

$$512 : m' :: 1.026 : 0.55,$$

and from this, by equating the products of the extreme and mean terms, we get

$$1.026m' = 281.6,$$

and finally, dividing by 1.026, we obtain

$$m' = \frac{281.6}{1.026} = 274.464 \text{ cubic inches.}$$

It therefore appears, that the quantity of the solid immersed below the plane of floatation, is 274.464 cubic inches; consequently, the part extant is $512 - 274.464 = 237.536$ cubic inches, being less than half the magnitude of the body, by 18.464 cubic inches.

313. EXAMPLE 2. Let the specific gravities and the magnitude of the body remain as in the last example; what weight must be added

to the body, in order that its upper surface may be made to coincide with that of the fluid?

Put $x =$ the weight which must be added to the solid, in order that it may sink to a level with the surface of the water; then, we have

$$m : m + x : : 0.55 : 1.026,$$

and by equating the products of the extremes and means, we get

$$0.55(m + x) = 1.026m;$$

therefore, by transposition, we obtain

$$0.55x = 0.476m;$$

but according to the question, $m = 512$ cubic inches, hence we get

$$0.55x = 243.712,$$

and finally, by division, we have

$$x = \frac{243.712}{0.55} = 443.113$$ cubic inches; but a cubic inch of fir of the given specific gravity, weighs 0.0198 lbs. avoirdupois, very nearly; consequently, the weight to be added for the purpose of making the solid sink to the same level as the surface of the fluid, is

$$.0198 \times 443.113 = 8.774 \text{ lbs. nearly.}$$

314. But to determine generally, the magnitude which must be added to the original solid, in order that its surface may be coincident with that of the fluid:—Let $x =$ the weight to be added; then, by the Proposition, we have

$$m + x : m : : s : s,$$

from which, by equating the products of the extremes and means, we get

$$s(m + x) = s'm,$$

and by separating the terms, it becomes

$$sm + sx = s'm,$$

and finally, by transposition and division, we obtain

$$x = \frac{m(s' - s)}{s}. \qquad (201).$$

Therefore, the practical method of reducing this equation, may be expressed in words in the following manner.

> RULE. *From the specific gravity of the fluid, subtract that of the solid; then, multiply the remainder by the magnitude of the solid, and divide the product by its specific gravity for the weight to be added.*

s 2

315. **Example** 3. A cubic mass of oak, whose specific gravity is 0.872, is placed in a cistern of water, and when it has attained a state of equilibrium with its sides vertical, it stands 20 inches above the surface of the fluid; what is the magnitude of the solid, the specific gravity of the water being represented by unity?

In order to resolve this example, we shall first investigate a general formula, which will apply to every case of a similar nature, when the specific gravity of the fluid and that of the solid are given; for which purpose,

Put $x =$ the length of one side of the solid; then is
$x - a$* $=$ the length of that portion which is below the plane of floatation.

But by the principles of mensuration, the magnitude of the whole body is

$$m = x^3,$$

and that of the part immersed, is

$$m' = (x - a) \times x^2 = x^3 - a x^2;$$

therefore, by the Proposition, we obtain

$$m : m' :: s' : s;$$

and this, by substitution, becomes

$$x^3 : x^3 - a x^2 :: s' : s;$$

from which by equating the products of the extremes and means, we get

$$s x^3 = s' (x^3 - a x^2);$$

and by separating the terms, we obtain

$$s x^3 = s' x^3 - a s' x^2;$$

consequently, by transposition and division, it is

$$x = \frac{a s'}{s' - s)}. \qquad (202).$$

And the practical rule supplied by this equation, may be expressed in words at length in the following manner.

> **Rule.** *Multiply the specific gravity of the fluid by the height of the body above its surface, and divide the product by the difference between the specific gravity of the fluid and that of the solid, and the quotient will give the side of the cube required.*

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* The quantity a, is here put to denote the height of the body above the fluid.

Taking, therefore, the data as proposed in the foregoing example, and we shall obtain

$$x=\frac{20\times 1}{1-0.872}=156.29 \text{ inches; consequently,}$$

the magnitude, or cubical contents of the body, is

$$156.29\times 156.29\times 156.29=3815627.7 \text{ inches.}$$

In addition to the foregoing examples, which might very appropriately have been ranked under the head of problems, the seventh proposition affords the following inferences.

316. INF. 1. If two bodies floating on the same fluid, be in a state of equilibrium :—

> *The specific gravities of those bodies, will be to one another directly as the parts below the plane of floatation, and inversely as the whole magnitudes of the bodies.*

317. INF. 2. If the same body float upon two different fluids, and be in a state of equilibrium on each :—

> *The specific gravities of the fluids, will be to one another, inversely as the parts of the body below the plane of floatation.*

318. INF. 3. If different bodies float in equilibrio on the surfaces of different fluids, and if the parts below the planes of floatation be equal among themselves :—

> *The specific gravities of the fluids, will be to one another directly as the weights of the bodies, or directly as the magnitudes of the bodies drawn into their specific gravities.*

319. INF. 4. If any number of bodies of the same weight, but of different specific gravities, float in equilibrio on the surface of the same fluid :—

> *The magnitudes of the parts below the plane of floatation, are equal to one another.*

320. INF. 5. If a body float in equilibrio on the surface of a given fluid, and if the part below the plane of floatation be increased or diminished by a given quantity :

> *The absolute weight of the body, (in order that the equilibrium may still obtain,) must be increased or decreased by a weight, which is equal to the weight of the portion of the fluid that is more or less displaced, in consequence of increasing or diminishing the immersed part of the body, or that which falls below the plane of floatation.*

321. The principle announced in the last inference, may be demonstrated in the following manner.

Put $m =$ the magnitude of the body at first, when in a state of equilibrium,
$m' =$ the part originally below the plane of floatation,
$m'' =$ the part by which it is increased or diminished,
$s =$ the specific gravity of the body,
$s' =$ the specific gravity of the fluid, and
$w =$ the weight by which the body is increased or diminished, in consequence of the increase or decrease of the immersed part.

Then, because the quantity of fluid displaced, is equal to the magnitude of the body which displaces it, it follows, that the weight of the displaced fluid is expressed by $(m' + m'')s'$, and the weight of the whole solid with which it is in equilibrio, is $(ms + w)$; consequently, we have

$$(m' + m'')s' = ms + w.$$

Now it is manifest, that in the case of the first equilibrium,

$$m's' = ms;$$

it therefore follows, that

$$m''s' = w.$$

That is, the weight by which that of the body must be increased or diminished, to restore the equilibrium :—

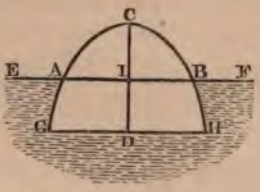
Is equal to the weight of that quantity of the fluid which is more or less displaced, in consequence of the increase or decrease of the part below the plane of floatation.

PROBLEM XLVI.

322. If a solid body in the form of a paraboloid, be in a state of quiescence on the surface of a fluid, whose specific gravity bears any relation to that of the body :—

It is required to determine how much of the solid will sink beneath the plane of floatation.

Let GACBH be a vertical section passing along the axis of the solid, and cutting the plane of floatation in the line AB; CD being the axis



of the solid, GH the diameter of its base, and ABHG the portion which falls below EF the surface of the fluid.

Put $a =$ CD, the axis of the paraboloid,
$d =$ GH, the diameter of its base,
$m =$ the magnitude of the entire solid GACBH,
$m' =$ the magnitude of the immersed part,
$s =$ the specific gravity of the body,
$s' =$ the specific gravity of the fluid, and
$x =$ CI, the axis of that portion of the body, which in a state of equilibrium, remains above the plane of floatation.

Consequently, by the seventh proposition, we have

$$m : m' :: s' : s.$$

But by the principles of mensuration, the solidity of a paraboloid is equal to one half the solidity of its circumscribing cylinder; therefore, we get

$$m = .3927ad^2,$$

and similarly, we obtain

$$\text{solidity ACB} = 1.5708px^2,$$

where p is the parameter of the generating parabola.

Now, the writers on conic sections have demonstrated, that according to the property of the generating curve,

$$4ap = d^2;$$

let this value of d^2 be substituted instead of it in the preceding value of m, and we shall obtain

$$m = 1.5708a^2p;$$

consequently, by subtraction, we get

$$m' = 1.5708p(a^2 - x^2);$$

therefore, by substituting these values of m and m' in the above analogy, it is

$$a^2 : a^2 - x^2 :: s' : s;$$

and from this, by equating the products of the extreme and mean terms, we obtain

$$s'a^2 - s'x^2 = sa^2,$$

and by transposition we have

$$s'x^2 = a^2(s' - s);$$

therefore, by division and evolution, we obtain

$$x = a\sqrt{\frac{s' - s}{s'}};$$

and finally, by subtraction, we have

$$\text{ID} = a\left(1 - \sqrt{\frac{s' - s}{s'}}\right). \qquad (203).$$

323. The practical rule for effecting the reduction of the above equation, may be expressed in words at length in the following manner.

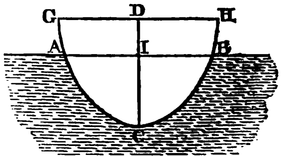
> Rule. *Divide the difference between the specific gravity of the fluid and that of the solid, by the specific gravity of the fluid; then, from unity subtract the square root of the quotient, and multiply the remainder by the axis of the parabola, and the result will give the height of the frustum that falls below the plane of floatation.*

324. Example. The axis of a paraboloid which floats in equilibrio on the surface of a fluid, is 29 inches; what part of the axis is immersed below the plane of floatation, supposing the body to be of oak, whose specific gravity is 0.76, that of water being unity?

Here, by proceeding as directed in the rule, we get

$$\text{ID} = 29\left(1 - \sqrt{\frac{1 - 0.76}{1}}\right) = 14.79 \text{ inches very nearly.}$$

If the vertex of the figure be downwards, as in the annexed diagram, then the part of the axis which falls below the plane of floatation will be greater than it is in the preceding case; for it is manifest, that since the same magnitude or part of the body must be immersed in both cases, it will require a greater portion of the axis, towards the vertex of the figure, to constitute that magnitude, than it would require towards the base.



Therefore, by retaining the foregoing notation, we have, by the principles of mensuration,

$$\text{GACBH} = m = 1.5708pa^2, \text{ and } \text{ACB} = m' = 1.5708px^2;$$

consequently, by the seventh proposition, we obtain

$$1.5708pa^2 : 1.5708px^2 :: s' : s;$$

therefore, by suppressing the common factors $1.5708p$, and equating the products of the extreme and mean terms, we get

$$s'x^2 = sa^2,$$

and this, by division, becomes

$$x^2 = \frac{s\,a^2}{s'};$$

and finally, by extracting the square root, we get

$$x = a\sqrt{\frac{s}{s'}}. \qquad (204).$$

325. The practical rule deduced from this equation is very simple; it may be expressed in words at length in the following manner.

> Rule. *Divide the specific gravity of the solid, by that of the fluid on which it floats; then, multiply the square root of the quotient by the axis of the body, and the product will give the height of the part below the plane of floatation.*

Therefore, by retaining the data of the foregoing example, we shall obtain as under,

$$\text{CI} = x = 29\sqrt{0.76} = 25.251 \text{ inches};$$

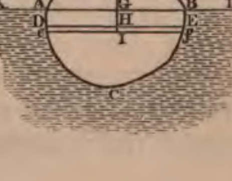
being a difference of 10.46 inches, in the depths of immersion, for the two cases of the question.

PROBLEM XLVII.

326. When a body floats in equilibrio on the surface of a homogeneous fluid, it is a necessary condition, that the centre of gravity of the body, and that of the fluid displaced, shall be in the same vertical line. Supposing, therefore, that the equilibrium is disturbed by the addition or subtraction of a certain weight:—

> *It is required to determine, how far the body will be depressed or elevated in consequence of the extraneous weight?*

The above problem will obviously admit of two cases, for a weight may be added to a body, and it may be subtracted from it; in the one case the body will descend, and in the other it will ascend; the following general solution, however, will answer both cases.



Let AFBC represent a vertical section of the solid body, floating in a state of equilibrium on a fluid of which the horizontal surface is KL; and suppose, that when the body is acted on by its own weight only, the straight line DE is coincident with the surface of the fluid; but in consequence of the additional weight *abcd*, the body descends through the space GH, where it again attains a state of quiescence, and the plane of floatation mounts to AB.

Now, it is manifest, that when the body is acted on by means of its own weight only, (in which case, DE is coincident with the surface of the fluid,) the weight of the whole body is equivalent to that of a quantity of the fluid, whose magnitude is DCE; but when the weight *abcd* is applied, the compound weight is equivalent to that of a quantity of the fluid, whose magnitude is ACB; consequently, the subsidiary weight *abcd*, and the weight of a quantity of the fluid, whose magnitude is ABED, are equal to one another.

Draw the straight line *ef* parallel and indefinitely near to DE; then is DE*fe*, the small elementary increase of the immersed portion of the body, corresponding to any indefinitely small increase of the weight *abcd*.

Put $a =$ the area of the horizontal section passing through AB, determinable from the position and the figure of the body, before the weight *abcd* is applied,

$w =$ the weight *abcd*,

$\dot{w} =$ the fluxion or small elementary variation of w;

$x =$ GH, the distance through which the body passes in consequence of the weight w being applied.

$\dot{x} =$ the fluxion or elementary variation of x, corresponding to $\dot{w}$, and

$s =$ the specific gravity of the fluid.

Then, because the line *ef* is supposed to be indefinitely near to DE, it follows, that the portion of the body whose section is DE*fe*, may be considered as uniform in area between its bases, and consequently, its magnitude is expressed by $a\dot{x}$; but DE*fe*, is equal to the quantity of fluid displaced by reason of the elementary weight $\dot{w}$, and it is a well attested principle in hydrostatics, that the weight of the quantity of fluid displaced, and that of the body which displaces it, are equal to one another; therefore we have

$$\dot{w} = as\dot{x},$$

and the aggregate of the small elementary weights, or the whole weight added, is

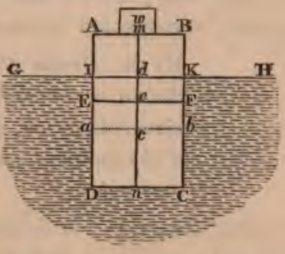
$$w = \int as\dot{x}. \qquad (205).$$

This is the general form of the equation of equilibrium; but it admits of various modifications, according to the conditions of the question and the nature of the body. For instance, if the body be a solid of revolution, and r the radius of the section coincident with the plane of floatation; then, the above equation becomes

$$w = \pi s \int r^2 \dot{x}, \qquad (206).$$

where the symbol π denotes the number 3.1416, or four times the area of a circle whose diameter is expressed by unity.

327. The solution of the problem, however, may be effected independently of the fluxional analysis, especially in all cases where the floating body is symmetrical with respect to its axis; for if it be in the form of a right cylinder with its axis vertical, as in the annexed diagram; then, the solution becomes an object of the greatest simplicity; for since the area of the horizontal section is constant, the space through which the body moves will be the same, whether the weight be added to it or subtracted from it.



Let ABCD be a vertical section of a cylinder, floating in equilibrio on a fluid whose surface is GH, the axis mn being perpendicular to the horizon, and suppose the weight w to be placed on the upper end of the cylinder; it is obvious that the equilibrium will then be destroyed, and the body will continue to descend, until it has displaced a quantity of the fluid, whose weight is equal to that of the compound mass, consisting of the cylinder, together with the applied body whose weight is w; or it will continue to descend, until the weight of the fluid displaced by the space IKFE is equal to w the weight of the applied body; in which case, the equilibrium will again obtain, and the plane of floatation, which originally cut the cylinder in EF, will now be transferred to IK.

Again, on the other hand, if a weight w be subtracted from the cylinder, supposed to be in a state of equilibrium with the plane of floatation passing through EF, the body will then ascend, until the weight of the fluid which rushes into its place becomes equal to the weight subtracted, in which case the solid will again be in a state of quiescence with the plane of floatation passing through ab.

Put $r =$ Id or Kd, the radius of the horizontal section,
$m =$ the magnitude of the space IKFE or EFba,
$x = de$ or ec, the space through which the body is depressed or elevated in consequence of the extraneous weight,

$w =$ the weight which is added to or subtracted from the cylinder, and

$s =$ the specific gravity of the fluid.

Then, by the principles of mensuration, the solidity of the cylinder, of which the section is IKFE or EFba, becomes

$$m = 3.1416 r^2 x,$$

and the weight of an equal magnitude of the fluid, is

$$ms = 3.1416 r^2 s x;$$

but this, by the nature of equilibrium, is equal to the disturbing weight; hence we have

$$w = 3.1416 r^2 s x,$$

and from this, by division, we obtain

$$x = \frac{w}{3.1416 r^2 s}. \qquad (207).$$

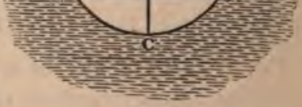
328. The practical rule for reducing this equation is very simple; it may be expressed in words at length in the following manner.

> RULE. *Divide the given disturbing weight, whether it be added to or subtracted from the cylinder, by the area of the horizontal section, drawn into the specific gravity of the fluid, and the quotient will express the quantity of descent or ascent accordingly.*

329. EXAMPLE. A cylinder of wood, whose diameter is 24 inches, floats in equilibrio with its axis vertical, on the surface of a fluid whose specific gravity is expressed by unity; now, supposing the equilibrium to be destroyed by the addition or subtraction of another body, of which the weight is 56 lbs.; through what space will the body move before the equilibrium be restored?

Here, by proceeding as directed in the rule, we have

$$x = \frac{56}{3.1416 \times 12^2 \times .03617^*} = 3.42 \text{ inches}.$$



Again, if the body should be in the form of a paraboloid, floating in equilibrio on the surface of a fluid with its vertex downwards, as represented in the annexed diagram, where ACB is a vertical section passing along the axis CD, and GH the surface of the fluid on

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[*] The decimal fraction 0.03617 expresses the weight in lbs. of a cubic inch of water.

which the body floats, with the plane of floatation originally passing through E F, but which, on the addition or subtraction of the weight w, ascends to I K or descends to $a\,b$.

Put $p =$ the parameter of the generating curve,
$\delta = e$C, the distance of the vertex below the surface of the fluid at first,
$m =$ the magnitude or solidity of the paraboloidal frustum, of which I K F E is a section,
$m' =$ the magnitude of the frustum whose section is E F $b\,a$;
$x = d\,e$, the descent of the body in consequence of the addition of the weight w,
$x' = e\,c$, the corresponding ascent in the case of subtraction,
and $s =$ the specific gravity of the fluid.

Then we have C$d = \delta + x$, and C$c = \delta - x'$, and according to the writers on mensuration, we have

$$m = 1.5708p(2\delta x + x^2),$$

and in a similar manner, we obtain

$$m' = 1.5708p(2\delta x' - x'2);$$

and since the weight of any body is expressed by its magnitude drawn into its specific gravity, it follows, that the weight of a quantity of fluid equal respectively to m and m', are

$$ms = 1.5708ps(2\delta x + x^2), \text{ and } m's = 1.5708ps(\delta x' - x'^2).$$

Now, these according to the conditions of the problem, are respectively equal to the disturbing weight; hence we have in the case of addition,

$$w = 1.5708ps(2\delta x + x^2), \qquad (208).$$

and in the case of subtraction, it is

$$w = 1.5708ps(2\delta x' - x'^2). \qquad (209).$$

330. The equations which we have just obtained, are precisely the same as would arise, by taking the fluent of the expression in equation (206); it therefore appears, that although the fluxional notation is the most convenient for expressing the general result, yet in point of simplicity as regards symmetrical bodies, there is little advantage to be derived from its adoption.

Suppose that in the first instance, the equilibrium is destroyed by the addition of the weight w, and let it be required to determine how far the body will descend in consequence of the addition.

Equation (208) involves this condition; consequently, if both sides be divided by the expression $1.5708ps$, we shall obtain

$$x^2+2\delta x=\frac{w}{1.5708ps},$$

which being reduced, gives

$$x=\sqrt{\delta^2+\frac{w}{1.5708ps}}-\delta. \qquad (210).$$

331. And the practical rule for reducing this equation, may be expressed in words at length, as follows.

> RULE. *Divide the weight which disturbs the equilibrium, by* 1.5708 *times the parameter of the generating parabola, drawn into the specific gravity of the fluid, and to the quotient add the square of the distance between the vertex of the figure and the plane of floatation in the first position of equilibrium; then, from the square root of the sum, subtract the said distance, and the remainder will express the quantity of descent.*

332. EXAMPLE. A solid body in the form of a paraboloid, floats on a vessel of water in a state of equilibrium with its vertex downwards, when 12 inches of the axis are immersed below the plane of floatation; how much farther will the body sink, supposing a weight of 28 lbs. to be laid on its base, the parameter of the generating parabola being 16 inches?

Here, by pursuing the directions of the rule, we get,

$$x=\sqrt{12^2+\frac{28}{1.5708\times16\times.03617}}-12=1.22 \text{ inches.}$$

333. If the weight w should be subtracted from the paraboloid instead of being added to it, the quantity of ascent will then be determined by equation (209), where we have

$$w=1.5708ps(2\delta x'-x'^2);$$

divide both sides of this equation by the quantity $1.5708ps$, and we shall obtain

$$2\delta x'-x'^2=\frac{w}{1.5708ps},$$

which, by transposing the terms, becomes

$$x'^2-2\delta x'=-\frac{w}{1.5708ps}.$$

By completing the square, we get

$$x'^2-2\delta x'+\delta^2=\delta^2-\frac{w}{1.5708ps},$$

and finally, by extracting the square root and transposing, it is

$$x' = \delta - \sqrt{\delta^2 - \frac{w}{1.5708ps}}. \qquad (211).$$

334. And the practical rule by which this equation is reduced, may be expressed in words at length, in the following manner.

> RULE. *Divide the weight which is proposed to be subtracted from the paraboloid, by* 1.5708 *times the parameter of the generating parabola drawn into the specific gravity of the fluid, and subtract the quotient from the square of the distance of the vertex below the plane of floatation in the first position of equilibrium; then, from the said distance, subtract the square root of the remainder for the quantity of ascent required.*

335. If we take the data of the foregoing example, and proceed according to the directions of the rule, we shall obtain

$$x' = 12 - \sqrt{144 - \frac{28}{1.5708 \times 16 \times .03617}} = 1.35 \text{ inches},$$

exceeding the descent in the former example, by 0.13 of an inch.

Numerous other examples akin to the above, respecting bodies of various forms, and placed in different positions, might here be proposed; but since they are all resolvable by the general formula, equation (206), we have thought proper to omit them.

PROBLEM XLVIII.

336. If a body which is symmetrical with respect to its vertical axis, floats upon a fluid in a state of equilibrium:—

> *It is required to determine what weight must be placed upon the body, so that it shall sink to a level with the surface of the fluid, the specific gravities of the solid and the fluid, together with the magnitude of the solid, being given.*

In the second example to the seventh proposition, we have advanced principles of nearly a similar import to those required for the solution of the present problem, yet nevertheless, we think a separate solution in the present instance is necessary, since it can be somewhat differently represented; for which purpose,

Put m = the entire magnitude of the floating body,

w = its weight before the external body or weight is superadded,

$w' =$ the added weight,
$w'' =$ the weight of the fluid displaced,
$s =$ the specific gravity of the fluid, and
$s' =$ the specific gravity of the floating solid.

Then, because the absolute weight of any body, is expressed by its magnitude drawn into its specific gravity; it follows, that the weight of the floating solid, is

$$w = ms',$$

and in like manner, the weight of the displaced fluid, is

$$w'' = ms;$$

now, it is manifest, from the nature of the problem, that the weight of the displaced fluid is equal to the weight of the floating body, together with the superadded weight; consequently, we have

$$w' + w = w' + ms' = ms;$$

from which, by transposition, we obtain

$$w' = m(s - s'). \qquad (212).$$

337. The practical rule for the reduction of this equation is very simple: it may be expressed as follows.

> Rule. *Multiply the difference between the specific gravities of the fluid and the floating solid, by the whole magnitude of the floating body, and the product will express the value of the added weight.*

338. Example. A mass of oak, whose specific gravity is .872, that of water being unity, floats in equilibrio on the surface of a fluid whose specific gravity is 1.038; what weight applied externally to the floating body, will depress it to the level of the fluid surface, supposing the magnitude of the body to be equal to 8 cubic feet?

Here, by operating as the rule directs, we shall have

$$w' = 8(1.038 - .872) = 1.328 \text{ cubic feet of water};$$

but it is a well known fact, that one cubic foot of water weighs $62\frac{1}{2}$ lbs. avoirdupois, very nearly; consequently, we have

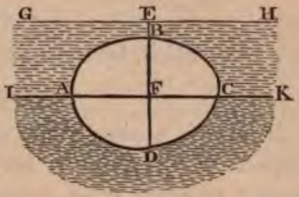
$$w' = 1.328 \times 62\tfrac{1}{2} = 83 \text{ lbs.}$$

PROPOSITION VIII.

339. If a solid body, which is specifically heavier than one of two fluids which do not mix, and specifically lighter than the other, be immersed in the fluids:—

It will float in equilibrio between them, when the weight of the fluids respectively displaced, are together equal to the weight of the solid body which causes the displacement; the specific gravities of the fluid being supposed known.

Let ABCD be a vertical section passing through the centre of gravity of the floating body, and suppose that IK is the common surface of the two fluids, in which the solid is quiescent, GH being the surface of the lighter fluid.



Now, it is manifest, that since the body ABCD is specifically heavier than one of the fluids, and specifically lighter than the other, it cannot be entirely at rest in either, but must rest between them in such a position, that the sum of the weights of the fluids displaced shall be equal to the whole weight of the solid.

Let EFD be perpendicular to IK, the common surface of the fluids in which the body floats; then it is evident, that the pressure downward on any point of the base D, is equal to the weight of the incumbent line of solid particles, whose altitude is BD the thickness of the body, together with the weight of EB the superincumbent column of the lighter fluid; and again, the pressure upwards on the same point D, is equal to the weight of a column of the heavier fluid whose altitude FD, together with the weight of a column of the lighter fluid, whose altitude is EF.

Put d = EB, the depth of the body below the upper surface of the lighter fluid,
d' = EF, the whole depth of the lighter fluid, or the depth of the common surface,
δ = FD, the depth of the body below the common surface, or the surface of the heavier fluid,
δ' = BD, the whole thickness of the solid body,
s = the specific gravity of the lighter fluid,
s' = the specific gravity of the heavier fluid,
s'' = the specific gravity of the solid body,
p = the downward pressure, and
p' = the upward pressure.

Then, because the weight of any body, whether it be fluid or solid, is expressed by the product of its magnitude drawn into its specific gravity, it follows that the downward pressure on the point D, is

$$p = \delta' s'' + ds,$$

and in like manner, the pressure upwards, is

$$p' = \delta s' + d's.$$

But when the body floats in a state of equilibrium, the upward and the downward pressures are equal to one another; hence we have

$$\delta' s'' + ds = \delta s' + d's,$$

from which, by transposing and collecting the terms, we get

$$\delta' s'' = \delta s' + (d' - d)s.$$

Now, it is obvious, that what we have demonstrated above with respect to the point D, may also be demonstrated to hold with respect to every other point of the surface whose section is ADC; consequently, by taking the aggregate of the upward and downward pressures, we obtain

$$(\delta' + \delta' + \delta' + \&c.)\,s'' = (\delta + \delta + \delta + \&c.)\,s' + \{(d' + d' + d' + \&c.) - (d + d + d + \&c.)\}\,s.$$

Put $m = (\delta' + \delta' + \delta' + \&c.)$ to infinity, equal to the magnitude of the entire floating body,

$m' = (\delta + \delta + \delta + \&c.)$ to infinity, equal to the part immersed in the heavier fluid, and

$m'' = \{(d' + d' + d' + \&c.) - (d + d + d + \&c.)\}$ to infinity, equal to the part immersed in the lighter fluid; consequently, by substitution, we get

$$m s'' = m' s' + m'' s. \qquad (213).$$

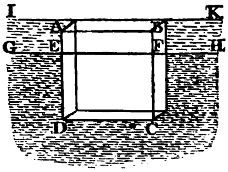
This equation involves the principle announced in the Proposition, and its application to practical cases will be exemplified in the resolution of the following Problems.

PROBLEM XLIX.

340. Suppose that a solid body in the form of a regular cube, is observed to float in equilibrio between two unmixable fluids of different specific gravities:—

It is required to determine, how much of each fluid is displaced by the body, the specific gravities of the body and the fluid being given.

Let ABCD be a cubical body, floating in equilibrio between the two unmixable fluids, whose upper horizontal surfaces are respec-



tively IK and GH, GH being the surface of contact of the two fluids.

Put $m = $ ABCD, the magnitude of the whole body,

$x = $ ABCFE, the magnitude of the part immersed in the lighter fluid, and

$m - x = $ EFCD, the magnitude of the part immersed in the heavier fluid.

Therefore, if the specific gravities of the body and the fluids be respectively denoted by s'', s and s', as in the Proposition; then, we shall have

$$ms'' = xs + (m - x)s',$$

from which, by transposition, we obtain

$$x(s' - s) = m(s' - s''),$$

and finally, by division, it becomes

$$x = \frac{m(s' - s'')}{(s' - s)}. \qquad (214).$$

341. An equation of an extremely simple and convenient form, from which we deduce the following practical rule.

> RULE. *Multiply the magnitude of the immersed solid by the difference between its specific gravity and that of the heavier fluid; then divide the product by the difference between the specific gravities of the fluids, and the quotient will give the magnitude of the part immersed in the lighter fluid.*

342. EXAMPLE. A cubical piece of oak containing 2000 inches, and whose specific gravity is 0.872, that of water being unity, floats in equilibrio between two fluids, whose specific gravities are respectively 1.24 and 0.716; what portion of the solid is immersed in each of the fluids, supposing them to be altogether unmixable?

The operation being performed according to the rule, will stand as below.

$$x = \frac{2000(1.24 - 0.872)}{(1.24 - 0.716)} = 1404.58 \text{ cubic inches.}$$

This result expresses the solid contents of that part of the body which is immersed in the lighter fluid; consequently, the part which is immersed in the heavier fluid, is

$$2000 - 1404.58 = 595.42 \text{ cubic inches.}$$

343. In this example, a greater portion of the body is immersed in the lighter fluid than what is immersed in the heavier; but this circumstance manifestly depends upon the nature of the immersed body, and the relation of the specific gravities; for an instance may readily be adduced, in which exactly the reverse conditions will obtain:— Thus, let the magnitude of the body and the specific gravities of the fluids remain as above, and suppose the specific gravity of the body to be 1.17 instead of 0.872; what then are the parts immersed in the respective fluids?

The numerical process is represented as below.

$$x=\frac{2000(1.24-1.17)}{(1.24-0.716)}=267.175 \text{ cubic inches.}$$

Here then, we have 267.175 cubic inches for the portion which is immersed in the lighter fluid, while that immersed in the heavier is $2000-267.175=1732.825$ cubic inches; this agrees with the case represented in the diagram, for there the body displaces a greater quantity of the heavier than it does of the lighter fluid.

PROBLEM L.

344. Having given the specific gravities of two unmixable fluids, and the magnitude of a solid body which floats in equilibrio between them:—

> *It is required to determine the specific gravity of the solid, so that any proposed part of it may be immersed in the lighter fluid.*

Put $m =$ the magnitude of the immersed solid, the same as in the preceding Problem,
$s =$ the specific gravity of the lighter fluid,
$s' =$ the specific gravity of the heavier fluid,
$x =$ the specific gravity of the solid body, being the required quantity, and
$n =$ the denominator of the fraction which expresses the part of the body immersed in the lighter fluid.

Then, according to the principle demonstrated in Proposition VIII., we shall obtain

$$mx=\frac{ms}{n}+\frac{(m \quad -m)s'}{n},$$

and from this, by a little farther reduction, we get

$$mnx=mns'-m(s'-s);$$

consequently, by division, we obtain

$$x = \frac{mns' - m(s' - s)}{mn}.\text{[*]} \qquad (215).$$

345. And from this equation we deduce the following rule.

RULE. *Multiply together, the magnitude of the body, the number which expresses what part of it is immersed in the lighter, and the specific gravity of the heavier fluid; then, from the product subtract the difference between the specific gravities of the fluid drawn into the magnitude of the solid body, and divide the remainder by the magnitude of the body, multiplied by the number which expresses what part of it is immersed in the lighter fluid; then shall the quotient express the specific gravity of the body.*

346. EXAMPLE. The specific gravities of two unmixable fluids are respectively 1.24 and 0.716, that of water being unity; now, supposing that when these fluids are poured into the same vessel, a body of 2000 cubic inches which is in equilibrio between them, has one seventh part of its magnitude immersed in the lighter fluid; what is the specific gravity of the body?

Here, by proceeding according to the rule, we have

$$mns' = 2000 \times 7 \times 1.24 = 17360$$

$$-m(s' - s) = 2000(1.24 - 0.716) = -1048 \text{ subtract}$$

$$\text{difference} = \overline{16312};$$

consequently, by division, we shall obtain

$$x = \frac{mns' - m(s' - s)}{mn} = \frac{16312}{2000 \times 7} = 1.17 \text{ nearly.}$$

347. From what has been done above, it is easy to ascertain what will be the specific gravity of the body, when equal portions of it are immersed in the lighter and in the heavier fluids; for in that case, we have n equal to 2, which being substituted in equation (215), gives

$$x = s' - \tfrac{1}{2}(s' - s). \qquad (216).$$

And the practical rule for reducing this equation may be expressed in words at length, in the following manner.

---

[*] This equation is susceptible of a simpler form, for by casting out the common factor m, it is

$$x = s' - \left(\frac{s' - s}{n}\right).$$

RULE. *From the specific gravity of the heavier fluid, subtract half the difference between the given specific gravities, and the remainder will be the specific gravity of the solid body.*

348. **EXAMPLE.** Let the specific gravities of the fluids remain as above; what must be the specific gravity of the body, so, that when it is in a state of equilibrium, one half of it may be immersed in each solid?

One half the difference of the given specific gravities, is

$$\tfrac{1}{2}(1.24-0.716)=0.262;$$

consequently, by subtraction, we have

$x=1.24-0.262=0.978$, and with this specific gravity, a body, whatever may be its magnitude, will be equally immersed in the two unmixable fluids.

349. If the specific gravity of the lighter fluid vanish, or become equal to nothing; then equation (215) becomes

$$ms''=m's',$$

and by converting this equation into an analogy, we get

$$m:m'::s':s''.$$

This analogy expresses the identical principle, which is announced and demonstrated in Proposition VII. preceding; it is therefore presumed, that the examples already given will be found sufficient to illustrate the application of this very elegant and important property.

Since the magnitude of the whole floating body is equal to the sum of its constituent parts, it follows, that according to our notation,

$$m=m'+m'';$$

consequently, by substitution, equation (215) becomes

$$(m'+m'')s''=m's'+m''s,$$

or by transposing and collecting the terms, we get

$$m''(s''-s)=m'(s-s''),$$

and by converting this equation into an analogy, we obtain

$$m':m''::(s''-s):(s'-s'').$$

By comparing the terms of the proportion as they now stand, it will readily appear, that if the specific gravity of the lighter fluid be increased, the term $(s''-s)$ is diminished, while $(s'-s'')$ remains the same; consequently, the first term m' will be diminished with respect to the second term m''; which implies, that the part of the body in the lighter fluid will be increased; hence arises the following very curious property, that

If any body float upon the surface of a fluid in vacuo, and air be admitted, the body will ascend higher above the surface, and consequently, the proportion of the immersed part to the whole will be diminished.

PROBLEM LI.

350. Suppose a solid body to float in equilibrio on the surface of water, both in air and in a vacuum :—

It is required to determine the ratio of the parts immersed in the water in both cases.

Put $m =$ the magnitude of the whole floating body,
$m' =$ the magnitude of the part immersed below the surface of the water, when the incumbent fluid is air,
$m'' =$ the portion immersed when the body floats on water in vacuo,
$s =$ the specific gravity of air,
$s' =$ the specific gravity of water, and
$s'' =$ the specific gravity of the floating body.

Then we have $m - m'$, for the part above the surface of the water, when the incumbent fluid is air, and $m - m''$ for the extant part when the body floats in vacuo; consequently, by equation (213), we have, when the body floats in air,

$$ms'' = m's' + (m - m')s,$$

from which, by a little reduction, we obtain

$$m'(s' - s) = m(s'' - s),$$

and finally, by division, it becomes

$$m' = \frac{m(s'' - s)}{(s' - s)}; \qquad (217).$$

and again, when the body floats in vacuo, we have

$$ms'' = m''s' + (m - m'')s;$$

but in this case, s vanishes, hence we get

$$ms'' = m''s',$$

and by division, it is

$$m'' = \frac{ms''}{s'}. \qquad (218).$$

Let the equations (217) and (218) be compared with one another in the terms of an analogy, and we shall have

$$m' : \frac{m(s'' - s)}{(s' - s)} :: m'' : \frac{ms''}{s'};$$

therefore, by equating the products of the extreme and mean terms, and casting out the common quantity m, we obtain

$$\frac{m''(s''-s)}{s'-s)}=\frac{m's''}{s'};$$

by clearing the equation of fractions, we get

$$m''s'(s''-s)=m's''(s'-s),$$

and finally, by division, we have

$$m''=\frac{m's''(s'-s)}{s'(s''-s)}. \qquad (219).$$

351. Now, it is manifest, that in order to determine from this equation, what part of the body is immersed in the water when it floats in vacuo, it is necessary in the first place, to ascertain how much of it is immersed when the floatation occurs in air:—Equation (217) determines this, and the practical rule deduced from the equations (217) and (219) may be expressed in words at length in the following manner.

> RULE. *From the specific gravity of the floating body, subtract the specific gravity of air; multiply the remainder by the magnitude of the body, and divide the product by the difference between the specific gravities of water and air, for the part which is immersed in water, when the incumbent fluid is air.*
>
> Again. *Multiply the difference between the specific gravities of water and air by the specific gravity of the floating body; divide the product by the difference between the specific gravities of the solid body and air, drawn into the specific gravity of water; then, multiply the quotient by the magnitude of the part immersed in water when the body floats in air, and the product will be the mugnitude of the part immersed in water, when the body floats in vacuo.*

352. EXAMPLE. A mass of oak whose specific gravity is 0.925, contains 185 cubic inches; what quantity of it exists below the plane of floatation, supposing it to float on water in vacuo, the specific gravity of the air being $0.001\dot{2}$ at the instant of observation?

By operating according to the directions given in the first clause of the rule, the quantity below the plane of floatation when the incumbent fluid is air, becomes

$$m'=\frac{185(0.925-0.001\dot{2})}{(1-0.001\dot{2})}=171.108 \text{ cubic inches};$$

therefore, according to the second clause of the rule, the part immersed when the body floats in vacuo, becomes

$$m'' = \frac{0.925(1 - 0.001\dot{2})}{1 \times (0.925 - 0.0012)} \times 171.108 = 171.125 \text{ cubic inches.}$$

353. If we refer to the equation (218) preceding, it will readily appear, that the above result may be determined with much less labour and greater simplicity; for the magnitude of the immersed part, when the body floats in vacuo, is there expressed in terms of the weight of the body and the specific gravity of water, and the practical rule for reducing the equation, may be expressed in words at length, as follows.

> RULE. *Multiply the magnitude of the body by its specific gravity; then divide the product by the specific gravity of water, and the quotient will express the magnitude of the immersed part when the body floats in vacuo.*

Therefore, by taking the data as proposed in the above example, the magnitude of the immersed part becomes

$$m'' = \frac{185 \times 0.925}{1} = 171.125 \text{ cubic inches;}$$ being precisely the same quantity as we obtained by the foregoing prolix operation.

If we compare the computed values of m' and m'' with one another, we shall find that the latter exceeds the former by a very small quantity, that is,

$$171.125 - 171.108 = 0.017,$$

which verifies the concluding inference under Problem L.

354. On the principles which we have explained and illustrated in the foregoing problems, depends the construction and application of the *Hydrometer*, an instrument which is generally employed for detecting and measuring the properties and effects of water and other fluids, such as their density, gravity, force, and velocity.

When the hydrometer is employed to determine the specific gravity of water, it is sometimes denominated an *aërometer* or *water-poise;* and being an instrument of very general utility in numerous philosophical experiments, we think it will not be amiss in this place, to discuss its nature and properties a little in detail; and we may here observe, that the following problems and remarks are quite sufficient to establish and exemplify its most important applications.

The hydrometer, or aërometer, in general consists of a long cylindrical stem of glass, or other metal, connected with two hollow balls, into the lower of which is introduced a small quantity of mercury or leaden shot, for the purpose of preventing the instrument from overturning, and causing it to float steadily in a vertical position, or perpendicularly to the surface of the fluid in which it is immersed.

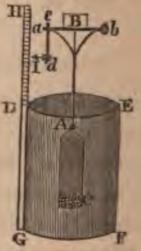
Numerous schemes have been promulgated by different ingenious and experienced philosophers for the improvement of this instrument; but however much the forms which have been recommended may differ among themselves, yet the general principle is the same in all.

The following is a list of the principal writers who have registered their improvements in the annals of science, viz.

Adie,
Atkins,
Brewster,
Charles,
Clark,
Deparcieux,
Desaguliers,
Dicas,
Fahrenheit,
Guyton,
Jones,
Levi,
Nicholson,
Quin,
Sikes,
Speer,
and
Wilson.

355. It would be quite superfluous to detail the various alterations and improvements suggested by these authors; suffice it to say, that in all there is something different and in all there is something common; but that which merits the greatest share of our attention, by reason of the extreme delicacy of its indications and the simplicity of its construction, is the hydrometer of Deparcieux, which was presented to the Academy of Sciences in the year 1766.

This instrument, which was intended by its inventor to measure the specific gravities of different kinds of water, is represented in the annexed figure, where A C is a glass phial about seven or eight inches in length, loaded with mercury or leaden shot, to prevent it from overturning; and in order that no air may lodge below it, when it is immersed in the fluid, the lower part is rounded off into the form of a spheric segment.



In the cork of the phial at A, is fixed a brass wire of one twelfth of an inch in diameter, and from thirty to thirty six inches long, or of any other length which may be found convenient for the purpose, but such, that when the phial is loaded and immersed in spring water of a medium temperature, the entire phial and about one inch of the wire should be below the graduated scale D H, which is fixed upon the side of the tin vessel D E F G; to the other end, or summit of the wire, is attached a small box B, intended for containing the minute weights which it may be found necessary to apply, in order to cause the instrument to sink to a certain fixed point in the different kinds of water whose specific gravities are required to be found.*

The white iron vessel DEFG, is used for holding the fluid on which the experiment is to be performed; it is generally about three feet in length, and from three to four inches in diameter, according to the circumstances under which it may happen to be employed. The small scale DH, is attached for the purpose of measuring the different depths to which the instrument sinks when differently loaded, or when it is immersed in fluids of different specific gravities.

The indications of this instrument are so extremely delicate, that if a small quantity of alcohol, or a little common salt, be added to the fluid, the phial will ascend or descend through a very sensible distance, which circumstance greatly enhances the value of the aërometer; for in proportion to its sensibility and the delicacy of its indications, are its importance and utility to be appreciated.

We come now to consider the theory of this instrument, and we shall just remark in passing, that the same principles, under very slight and obvious modifications, will apply to any other hydrometric instrument, of a similar, or nearly similar nature and construction, to that which forms the subject of our present discussion.

PROBLEM LII.

356. Having given the capacity or volume of the phial, together with the dimensions of the immersed wire, and the entire weight of the aërometer:—

It is required to determine the specific gravity of the fluid, in which the instrument settles in a state of equilibrium.

Now, because the weight of any body when floating in equilibrio, whatever may be its form and the substance of which it is composed, is equal to the weight of the fluid which it displaces; it follows, that if we put $c =$ the capacity or volume of the phial immersed in the fluid,

$l =$ the length of the immersed wire,

$r =$ the radius of its transverse section,

$\pi = 3.1416$, the number which expresses the circumference of a circle whose diameter is equal to unity,

---

* To the bottom of the box B we have affixed the arm ab, from one extremity of which is suspended the wire cd carrying the index I, the whole being truly balanced by the small ball b attached to the other extremity of the horizontal arm ab. In all other respects the instrument is that of Deparcieux.

$s =$ the specific gravity of the fluid sought, and
$w =$ the entire weight of the aërometer, always known.

Then, it is manifest, that the capacity or volume of the phial, together with the magnitude of the immersed wire, is equal to the quantity of fluid displaced; and the weight of this quantity of fluid is equal to the weight of the aërometer; but by the principles of mensuration, the magnitude of the immersed wire is expressed by $\pi r^2 l$; consequently, the quantity of fluid displaced is $c + \pi r^2 l$, and the magnitude of any body multiplied by its specific gravity is equal to its weight; hence we have

$$w = (c + \pi r^2 l)s; \qquad (220).$$

therefore, by division, we obtain

$$s = \frac{w}{c + \pi r^2 l}. \qquad (221).$$

357. Here follows the practical rule for reducing the equation.

> RULE. *Divide the entire given weight of the aërometer, by the capacity or volume of the phial, increased by the quantity of wire immersed, and the quotient will give the specific gravity of the fluid.*

358. EXAMPLE. The whole weight of an aërometer, when so loaded as to have the attached wire depressed 15 inches below the surface of the fluid, is 23 ounces; required the specific gravity of the fluid, supposing the diameter of the wire to be one twelfth of an inch, and the capacity of the phial 40 inches?

Here, by the mensuration of solids, the magnitude of the wire is

$$15 \times 3.1416 \times \tfrac{1}{24} \times \tfrac{1}{24} = 0.082 \text{ of a cubic inch, very nearly;}$$

therefore, the whole quantity of fluid displaced, is

$$40 + 0.082 = 40.082 \text{ cubic inches;}$$

therefore, by the rule, we obtain

$$s = \frac{23}{40.082} = 0.5738.$$

The number 0.5738, which we have obtained from the above calculation, expresses the weight of one cubic inch of the fluid in ounces; but since it is customary to express the specific gravity of bodies in ounces per cubic foot, it becomes necessary, for the sake of comparison, to reduce the above result to that standard; hence we have

$s = 0.5738 \times 1728 = 991.5264$ ounces per cubic foot for the specific gravity of the fluid on which the experiment was tried.

PROBLEM LIII.

359. Having given the capacity or volume of the phial, the whole weight of the aërometer, the specific gravity of the fluid, and the radius of the wire:—

> *It is from thence required to determine, how much of the stem or wire is immersed below the surface of the fluid when the instrument rests in a state of equilibrium.*

By recurring to the equation marked (220), and separating the terms, we obtain

$$\pi r^2 s l = w - c s;$$

from which, by division, we get

$$l = \frac{w - c s}{\pi r^2 s}. \qquad (222).$$

360. The practical rule for reducing this equation, may be expressed in words at length, in the following manner.

> RULE. *From the entire weight of the hydrometer, subtract the capacity of the phial drawn into the specific gravity of the fluid; then, divide the remainder by the area of a transverse section of the wire, drawn into the specific gravity of the fluid, and the quotient will express how far the wire is immersed below the upper surface of the fluid, when the instrument floats in a state of equilibrium.*

361. EXAMPLE. The entire weight of an aërometer, when so adjusted as to remain at rest in a fluid whose specific gravity is 0.5738* $=23$ ounces; what length of the stem or upright wire falls below the surface of the fluid, supposing its diameter to be one twelfth of an inch, and the magnitude of the immersed phial 40 inches?

Here, by the foregoing rule, we have

$$l = \frac{23 - 40 \times 0.5738}{3.1416 \times \frac{1}{24} \times \frac{1}{24} \times 0.5738} = 15.33 \text{ inches nearly.}$$

362. If the entire weight of the aërometer be multiplied by 1728, the number of cubic inches in one cubic foot, the formulas (221) and (222) become transformed into

---

* The number 0.5738, by which the specific gravity is here expressed, is the weight in ounces of one cubic inch, which being reduced to the standard of one cubic foot, gives $s = 0.5738 \times 1728 = 981.5624$ oz.

$$s = \frac{1728w}{c + \pi r^2 l}, \text{ and } l = \frac{1728w - cs}{\pi r^2 s};$$

from the first of which the standard specific gravity is obtained, and in the second, the specific gravity as calculated from the first must be employed.

By comparing the quantities in equation (222) with each other, it will readily be perceived, that a very small variation in w the weight of the instrument, or in s the specific gravity of the fluid, will produce a very considerable variation in l, the immersed portion of the stem or wire; for it is manifest, that the numerator of the fraction $w - cs$, expresses the weight of the fluid displaced by the wire or upright stem of the instrument, and consequently, since r the radius of the stem is a very small quantity, it follows, that the weight of the fluid which it displaces must also be very small.

PROBLEM LIV.

363. Suppose that a small variation takes place in the density of the fluid in which the instrument is immersed:—

> *It is required to determine the corresponding variation that takes place in the depth to which it sinks before the equilibrium is restored.*

Let the notation for the first position of equilibrium remain as in Problem LII., and let l' denote the immersed length of the stem or wire, corresponding to the specific gravity s'; then, by equation (222), we have

$$l' = \frac{w - cs'}{\pi r^2 s'};$$

consequently, by subtraction, the variation in length becomes

$$l - l' = \frac{w - cs}{\pi r^2 s} - \frac{w - cs'}{\pi r^2 s'},$$

and this, by a little farther reduction, gives

$$l - l' = \frac{w(s' - s)}{\pi r^2 s s'}. \qquad (223).$$

364. The practical rule for reducing this equation, may be expressed in words as follows.

> RULE. *Multiply the whole weight of the aërometer by the variation in the specific gravity; then, divide the product by the area of the transverse section of the upright stem or wire,*

drawn into the greater and lesser specific gravities of the fluid, and the product will express the required variation in the position of the instrument.

365. EXAMPLE. Suppose the specific gravity of the fluid to vary from 0.5738 to 0.5926 ounces per cubic inch during the time of the experiment, what is the corresponding variation in the depth of the instrument, its whole weight being 23 ounces, and the diameter of the upright stem equal to one twelfth of an inch?

Here, by attending to the directions in the rule, we obtain

$$l - l' = \frac{23(0.5926 - 0.5738)}{3.1416 \times 0.5926 \times 0.5738} \times 576 = 233.155 \text{ inches.}$$

Hence it appears, that by a difference of 0.0325 in the absolute specific gravity of the fluid, there arises a difference of 233 inches in the position of the instrument; this seems a very great difference, and is in reality far beyond the bounds prescribed for the whole apparatus to occupy; it serves, however, to exemplify the extreme delicacy of the principle, and when the changes in the specific gravity are very minute, the corresponding changes in depth will nevertheless be sufficiently distinct to admit of an accurate measurement.

366. By diminishing the diameter of the upright stem, or increasing the entire weight of the instrument, which is equivalent to an increase in the weight of the fluid displaced, the sensibility of the aërometer may be greatly increased. This is manifest, for by inference 5, equation (202), it will readily appear, that if the specific gravity remains the same, the quantity by which the instrument sinks in the fluid on the addition of a small weight w', varies directly as the magnitude of the weight added, and inversely as the square of the radius of the upright stem.

Let us suppose, that by the addition of the small weight w', the length of the part of the stem l, which is originally immersed, becomes equal to l'; then, by the principles of mensuration, the increased magnitude of the immersed stem is $\pi r^2(l' - l)$; but the weight of a body is equal to its magnitude multiplied by its specific gravity; hence we have

$$\pi r^2(l' - l)s = w';$$

and this, by division, becomes

$$l' - l' = \frac{w'}{\pi r^2 s}.$$

Now it is obvious, that by the supposition of a constant specific gravity, the quantity πs is also constant: it therefore follows, that

$$l' - l \text{ varies as } \frac{w'}{r^2}.$$

In the above investigation, we have supposed the specific gravity of the fluid to remain constant; but admitting it to vary, so that s may become equal to s'; then, in order that the upright stem may rest at the same depth of immersion, w must become equal to $(w + w')$; if, therefore, we substitute s and $(w + w')$, for s and w in equation (223), we shall obtain

$$l = \frac{w + w' - cs'}{\pi r^2 s'},$$

an equation from which we find the value of s' to be

$$s' = \frac{w - w'}{c + \pi r^2 l},$$

and by a similar reduction, equation (223) gives

$$s = \frac{w}{c + \pi r^2 l};$$

consequently, by analogy, and suppressing the common denominator, we get

$$s' : s :: w + w' : w.$$

From this analogy, the difference between the specific gravities in the two cases can very easily be ascertained, for by the division of ratios, we have

$$s' - s : s :: w + w' - w : w,$$

which, by reduction, becomes

$$s' - s = \frac{w' s}{w}. \qquad (224).$$

367. This is a very simple equation for expressing the difference of the specific gravities; it may be reduced by the following practical rule.

> Rule. *Multiply the added weight by the lesser specific gravity; then, divide the product by the lesser weight, and the quotient will be the difference between the specific gravities sought.*

368. Example. An aërometer, whose absolute weight is equal to 23 ounces, is quiescent in a fluid whose specific gravity is 0.5738 ounces, as referred to a cubic inch; but on being put into a denser fluid, it requires the addition of 0.7536 of an ounce, to cause the instrument to sink to the same depth; what is the specific gravity of the denser fluid?

Here then we have given $w' = 0.7536$ of an ounce, and $s = 0.5738$; consequently, by the above rule, we have

$$s' - s = \frac{0.7536 \times 0.5738}{23} = .0188;$$

consequently, the specific gravity of the heavier fluid, is

$$s' = 0.5738 + 0.0188 = 0.5926;$$

and this, when reduced to the standard of one cubic foot, becomes $0.5926 \times 1728 = 1024.0128$, which, on being referred to a table of specific gravities, will be found to correspond with sea water at a medium temperature.

In the above operation, we have taken the specific gravity as referred to one cubic inch of the fluid only, but the well informed reader will readily perceive, that the same result would obtain if the specific gravity should be estimated by the cubic foot; for in that case, we should have $w' = 0.7536$ of an ounce, and $s = 991.5264$, consequently, by the rule, we have

$$s' - s = \frac{991.5264 \times 0.7536}{23} = 32.4864;$$

therefore by transposition, the specific gravity of the denser fluid, is $s' = 991.5264 + 32.4864 = 1024.0128$, being precisely the same result as that which we obtained on the former supposition.

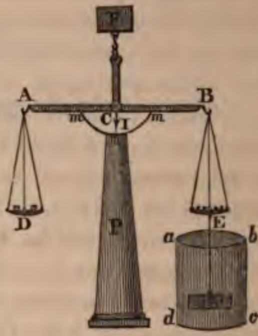
369. The diagram which we have employed to illustrate the general principle of the aërometer, is at the best but a very rude and imperfect representation, and in its present state, it is altogether unfitted for ascertaining the specific gravities of fluids with any degree of precision; it is therefore requisite, in cases where extreme accuracy is required, to have recourse to some other method of indicating the precise measure of density, and for this purpose, the hydrometer or aërometer, is very advantageously replaced by the

HYDROSTATIC BALANCE,

an instrument which determines the specific gravities of bodies with the greatest correctness, and which, on account of its simplicity and cheapness, is rendered available for almost every purpose in which the specific gravity of bodies forms the subject of inquiry.

The *Hydrostatical Balance*, so called, is nothing more than a common balance, furnished with some additional apparatus for enabling it to measure the specific gravities of bodies with accuracy and expedition, whether the bodies be in a solid or a fluid state. The description of the instrument is as follows.

Let A B be the beam of a balance very nicely equipoised upon its centre of motion at C, and suspended from the fixed object represented at F, the centering being so delicately executed, that the equilibrium of the instrument is disturbed by the smallest portion of a grain being added to or subtracted from either arm of the beam.



D and E are two scales, which, together with their appendages are also balanced with the greatest exactness; one of them as E having a hook in the middle of its bottom surface, to which the weight *w* is suspended by means of a horse hair, or any other flexible substance of such extreme levity, as to have no sensible effect upon the equilibrium.

P is an upright pillar placed directly under the centre of motion, and carrying the circular arc *m m m*, which serves to prevent a too great vibration on either side, and also, by means of the index I, which is fixed on the beam immediately under the fulcrum, it indicates the exact position of equilibrium; for it is manifest, that when the beam is horizontal, the pointer must be directly over the middle of the arc.

The pieces in the scale D, denote the weight of the body when weighed in air; but when the body is immersed in water, as represented by the figure *a b c d*, the scale D with its accompanying weights, must evidently preponderate, and for the purpose of restoring the equilibrium, small weights must be placed in the opposite scale at E; and since the weights thus added, indicate the weight of a quantity of water of equal bulk with the immersed body, it follows, that the specific gravity of the body can from thence be determined.

The hydrostatical balance, like the hydrometer or aërometer previously explained, has undergone various alterations and improvements, according to the ideas of the different individuals who have had occasion to apply it in their inquiries respecting the specific gravities of bodies; but since the general principle is the same in all, under whatever form the instrument may appear, it would lead to nothing useful to enter into a detailed description of the various improvements which it has received, and the numerous changes that have been made upon it; we shall therefore refrain from farther discussion on the nature of its construction, and proceed to exemplify the manner in which it is applied to the determination of specific gravities.

PROBLEM LV.

370. Having given the specific gravity of distilled water, equal to 1000 ounces per cubic foot:—

It is required to determine the specific gravity of a solid body that is wholly immersed in it.

It is manifestly implied by the total immersion of the body, that its specific gravity exceeds the specific gravity of the fluid in which it is immersed; therefore, attach the body to the hook in the bottom of the scale E by a very fine and light thread, and balance it exactly by weights put into the other scale at D; then, immerse the body in the water, and find what weight is required to restore the equilibrium, the weight thus required will measure the specific gravity of the body.

Put $w =$ the weight of the body when weighed in water,
$w' =$ the weight when weighed in atmospheric air,
$s =$ the specific gravity of water, and
$s' =$ the specific gravity of the body sought.

Then is $w - w'$ equal to the weight which must be put into the scale E to restore the equilibrium; consequently, by the fifth proposition, we have

$$w' - w : w' :: s : s';$$

from which, by reduction, we get

$$s' = \frac{w's}{w' - w}. \qquad (225).$$

371. The following is the practical rule in words at length for reducing the above equation.

RULE. *Multiply the weight of the body when weighed in air, by the specific gravity of the fluid, and divide the product by the weight which it loses in water for the specific gravity of the body.*

This rule determines the specific gravity of the body when it exceeds that of the fluid in which it is weighed; but when the body is specifically lighter than the fluid, the method of finding its specific gravity is shown in Problem XLIV., it is therefore unnecessary to repeat it here.

372. EXAMPLE. If a piece of stone weighs 20 lbs. in air, but in water only $13\frac{1}{2}$ lbs.; required its specific gravity, that of water being 1000?

Here, by the rule, $w = 13\frac{1}{2}$, $w' = 20$, $s = 1000$,

therefore $s' = \frac{20 \times 1000}{20 - 13\frac{1}{2}} = \frac{20000}{6.5} = 3076.923 =$ the specific gravity of the mass when it is wholly immersed in water.

CHAPTER XII.

OF THE POSITIONS OF EQUILIBRIUM.

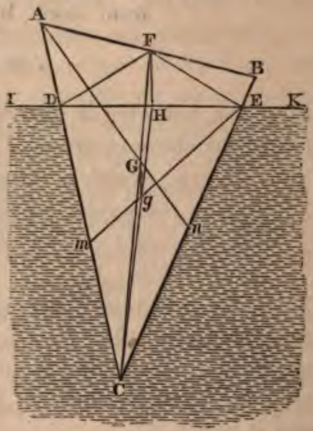
PROBLEM LVI.

373. Suppose that a solid homogeneous triangular prism, floats upon the surface of a fluid of greater specific gravity than itself, with only one of its edges immersed:—

It is required to determine in what position it will rest, when it has attained a state of perfect equilibrium.

Let ABC be a vertical transverse section, at right angles to the axis of the homogeneous prism, floating in a state of equilibrium on the fluid whose horizontal surface is IK.

Bisect AB, BC the sides of the triangle in the points F and n, and DE, DC in the points H and m; draw the straight lines CF and An, intersecting one another in the point G, and CH, Em intersecting in g; then is G the centre of gravity of the whole triangle ABC, and g the centre of gravity of the triangle DEC, which falls below DE the plane of floatation.



Join the points G, g by the straight line Gg; then, according to the principle announced and demonstrated in the sixth proposition, the straight line gG is perpendicular to DE the surface of the fluid.

Draw FH, and because CF and CH the sides of the triangle CFH, are cut proportionally in the points G and g, it follows from the principles of geometry, that the straight lines Gg and FH are parallel to one another; but we have shown that gG is perpendicular to the horizontal surface of the fluid, or the plane of floatation passing through DE; consequently, FH is also perpendicular to DE, and FD, FE are equal to one another.

Put $a =$ AB, the unimmersed side of the triangular section,
$b =$ BC, one of the sides which penetrate the fluid,
$c =$ AC, the other penetrating side,
$d =$ CF, the distance between the vertex of the section, and the middle of the extant side,
$\phi =$ the angle ACF, contained between the side AC and the line CF,
$\phi' =$ the angle BCF, contained between the line CF and the side B,
$s =$ the specific gravity of the solid body,
$s' =$ the specific gravity of the fluid on which it floats,
$x =$ CD, the immersed portion of the side AC, and
$y =$ CE, the immersed portion of the side BC.

Then, according to the principles of geometry, since the line CF is drawn from the vertex of the triangle at C, to the middle of the base or opposite side at F, it follows, that

$$\mathrm{AC}^2 + \mathrm{BC}^2 = 2(\mathrm{AF}^2 + \mathrm{CF}^2),$$

or by taking the symbolical representatives, we shall obtain

$$b^2 + c^2 = 2(\tfrac{1}{4}a^2 + d^2);$$

from which, by reduction, we get

$$d = \tfrac{1}{2}\sqrt{2(b^2 + c^2) - a^2}. \qquad (226).$$

Since all straight lines drawn parallel to the axis of the prism are equal among themselves; it follows, that the weight of the whole solid ABC, and that of the portion DEC below the plane of floatation, which corresponds to the magnitude of the fluid displaced, are very appropriately represented by the areas drawn into the respective specific gravities of the solid and the fluid on which it floats.

Now, the writers on the principles of mensuration have demonstrated, that the area of any right lined triangle:—

> *Is equal to the product of any two of its sides, drawn into half the natural sine of their contained angle.*

Therefore, if we put a' and a'' to represent the areas of the triangles ABC and DEC respectively, we shall have for the area of the triangle ABC,

$$a' = \tfrac{1}{2}bc\sin.(\phi + \phi'),$$

and for the area of the triangle DEC, it is

$$a'' = \tfrac{1}{2}xy\sin.(\phi + \phi').$$

But according to the principle demonstrated in the third proposition preceding, the weight of a floating body :—

Is equal to the weight of the quantity of fluid displaced.

Consequently, the weight of the solid prism whose section is ABC, is equal to the weight of the fluid prism, whose section is DEC; that is,

$$\tfrac{1}{2}bcs\sin.(\phi + \phi') = \tfrac{1}{2}xys'\sin.(\phi + \phi'),$$

and from this, by suppressing the common quantities, we get

$$bcs = xys'. \qquad (227).$$

By the principles of Plane Trigonometry, it is

$$\text{FD}^2 = d^2 + x^2 - 2dx\cos.\phi, \text{ and } \text{FE}^2 = d^2 + y^2 - 2dy\cos.\phi';$$

but these by construction are equal; hence we have

$$x^2 - 2dx\cos.\phi = y^2 - 2dy\cos.\phi'.$$

Let the value of d as expressed in equation (226), be substituted instead of it in the above equation, and we shall obtain

$$x^2 - x\cos.\phi\sqrt{2(c^2 + b^2) - a^2} = y^2 - y\cos.\phi'\sqrt{2(c^2 + b^2) - a^2}. \quad (228).$$

Recurring to equation (227), by division, we have

$$y = \frac{bcs}{xs'}, \text{ the square of which is } y^2 = \frac{b^2c^2s^2}{x^2s'^2};$$

substitute these values of y and y^2 in equation (228), and it is

$$x^2 - x\cos.\phi\sqrt{2(c^2 + b^2) - a^2} = \frac{b^2c^2s^2}{x^2s'^2} - \frac{bcs\cos.\phi'}{xs'}\sqrt{2(c^2 + b^2) - a^2},$$

and multiplying by x^2 we obtain,

$$x^4 - \cos.\phi\sqrt{2(b^2 + c^2) - a^2} \times x^3 = \frac{b^2c^2s^2}{s'^2} - \frac{bcs\cos.\phi'}{s'}\sqrt{2(b^2 + c^2) - a^2} \times x,$$

from which, by transposition, we get

$$x^4 - \cos.\phi\sqrt{2(b^2 + c^2) - a^2} \times x^3 + \frac{bcs\cos.\phi'}{s'}\sqrt{2(b^2 + c^2) - a^2} \times x = \frac{b^2c^2s^2}{s'^2}. \qquad (229).$$

374. The equation as we have now exhibited it, involves the several circumstances that accompany the equilibrium of a floating body, and its root determines the position in which the equilibrium obtains; the general form of the expression, is however exceedingly complex, and rising as it does to the fourth order or degree, the resolution is necessarily attended with considerable difficulty, especially when the sides

of the transverse section are represented by large numbers; in particular cases, the ultimate form will admit of being modified, and may in consequence, be rendered somewhat more simple; but it must nevertheless be understood, that whenever the position of equilibrium is required by computation, we must inevitably perform a very irksome and laborious process.

A geometrical construction may also be effected by the intersection of two hyperbolas; but since this implies a knowledge of principles higher than elementary, we think proper to pass it over, and proceed to illustrate the application of the above equation by the resolution of a numerical example.

375. EXAMPLE. Suppose a triangular prism of *Mar Forest fir*, the sides of whose transverse section are respectively equal to 28, 26, and 18 inches, to float in equilibrio in a cistern or reservoir of water, having only one angle immersed; it is required to determine the position of equilibrium, on the supposition that the two longest sides of the section penetrate the fluid, the specific gravity of the prism being to that of water as 686 to 1000?

By recurring to equation (229), and comparing its several constituent quantities with the parts of the diagram to which they respectively refer, it will readily appear, that x, $\cos.\phi$ and $\cos.\phi'$ are the only terms whose values require to be calculated; of which $\cos.\phi$ and $\cos.\phi'$ are to be determined from the nature of the figure, and x from the resolution of the biquadratic equation in which its values are involved.

The length of the straight line CF, which is drawn from the vertex of the section at C, to the middle of the opposite side at F, is according to equation (226), expressed by

$$d = \tfrac{1}{2}\sqrt{2(b^2+c^2)-a^2};$$

consequently, by substituting the numerical values of the sides, we obtain

$$d = \tfrac{1}{2}\sqrt{2(28^2+26^2)-18^2} = 25.4754784 \text{ inches.}$$

Hence, in the triangles ACF and BCF respectively, we have given the three sides AC, AF, FC and BC, BF, FC to find cos.ACF and cos. BCF; for which purpose, we have the following equations as deduced from the elements of Plane Trigonometry, viz. In the triangle ACF, it is

$$\cos.\phi = \frac{4(c^2+d^2)-a^2}{8cd};$$

and in the triangle BCF, it is

$$\cos.\phi' = \frac{4(b^2+d^2)-a^2}{8bd}.$$

Now, by substituting the numerical values of a, b and c, as given in the question, and the value of d as deduced from calculation, the absolute values of cos.ϕ and cos.ϕ' will stand as below.

Thus, for the absolute numerical value of cos.ϕ, we have

$$\cos.\phi = \frac{4(784+649)-324}{8\times 28\times 25.4754784} = 0.94769,$$

and for the absolute numerical value of cos.ϕ', it is

$$\cos.\phi' = \frac{4(676+649)-324}{8\times 26\times 25.4754784} = 0.93906.$$

Let the numerical values of cos.ϕ and cos.ϕ' as determined by the above computation, together with the numerical values of a, b, c, s, and s', as given in the question, be respectively substituted in equation (229), and we shall obtain

$$x^4 - 48.2859x^3 + 23894.7x = 249408;$$

but in order to simplify the resolution of this equation, it will suffice to take the co-efficients to the nearest integer, for the error thence arising will be of very little consequence in cases of practice, and the modification will very much abridge the labour of reduction; the equation thus altered, will stand as below.

$$x^4 - 48x^3 + 23895x = 249408.$$

Therefore, if this equation be reduced by the method of approximation, or otherwise, the value of x will come out a very small quantity less than 22 inches; but taking it equal to 22, the result of the equation is

$$22^4 - 48\times 22^3 + 23895\times 22 = 248842.$$

By substituting the given values of b, c, s and s', with the computed value of x, in equation (227), we shall have

$$22000y = 499408,$$

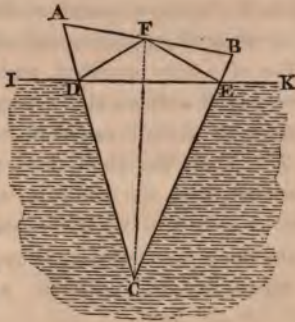
from which, by division, we obtain

$$y = \frac{499408}{22000} = 22.7 \text{ inches.}$$

Consequently, from these computed dimensions, together with the sides of the section given in the question, the prism may be exhibited in the position which it assumes when floating in a state of equilibrium.

376. Construct the triangle ABC to represent the transverse section of the floating prism, and such that the sides AC, BC, and AB are respectively equal to 28, 26, and 18 inches; make CD and CE respectively

equal to 22 and 22.7 inches, and through the points D and E, draw the straight line IK, which will coincide with the plane of floatation, or the surface of the fluid on which the body floats.



Bisect AB, the extant side of the section in the point F, and join FD and FE; then, the conditions of equilibrium manifestly are, that the lines FD and FE are equal to one another, and that the area of the immersed triangle DCE, is to the area of the whole triangle ACB, as the specific gravity of the solid is to the specific gravity of the fluid.

That the lines FD and FE are equal to one another, appears from an inspection and measurement of the figure; but the following proof by calculation will be more satisfactory, inasmuch as numbers can be more correctly estimated than measured lines, which depend for their accuracy upon the delicacy of the instruments and the address of the operator.

In the plane triangle DCF, we have given the two sides DC and CF, respectively equal to 22 and 25.4754784 inches, and the natural cosine of the contained angle DCF equal to 0.94769; consequently, the third side DF can easily be found; for by the principles of Plane Trigonometry, we know that

$$DF^2 = DC^2 + FC^2 - 2DC.FC \cos.DCF;$$

therefore, by substituting the respective numerical values, we obtain

$$DF^2 = 484 + 649 - 2 \times 22 \times 25.4754784 \times 0.94769 = 70.72;$$

consequently, by extracting the square root, it is

$$DF = \sqrt{70.72} = 8.4 \text{ inches.}$$

Again, in the plane triangle ECF, we have given the two sides EC and CF, respectively equal to 22.7 and 25.4754784 inches, and the natural cosine of the contained angle ECF equal to 0.93906; consequently, by Plane Trigonometry, we have

$$EF^2 = EC^2 + CF^2 - 2EC.CF \cos.ECF;$$

and substituting the respective numerical values, we obtain

$$EF^2 = 515.29 + 649 - 2 \times 22.7 \times 25.4754784 \times 0.93906 = 77.895;$$

therefore, by extracting the square root, we shall have

$$EF = \sqrt{77.895} = 8.82 \text{ inches.}$$

Hence, the length of the line DF is 8.4 inches, and the length of EF is 8.82 inches, giving a difference of 0.42, or something less than half an inch; being as small a difference as could be expected, from the manner in which the co-efficients of the equation that furnished the value of x were modified, and also from the circumstance of x being determined only to the nearest integer, without considering the fractions with which it might be affected.

377. Upon the whole then, the position of equilibrium is sufficiently manifest, from the condition of equality between the straight lines DF and EF; we shall therefore proceed to inquire if it be equally apparent, from the proportionality between the triangles ACB and DCE.

Since $\cos.\phi = 0.94769$, and $\cos.\phi' = 0.93906$, it follows that $\phi = 18^\circ\ 37'$ and $\phi' = 20^\circ\ 6'$; consequently, by addition, the whole angle ACB becomes

$$\phi + \phi' = 18^\circ\ 37' + 20^\circ\ 6' = 38^\circ\ 43';$$

therefore, in each of the triangles ACB and DCE, we have given the two sides AC, BC and DC, EC with the contained angle ACB common to both, to find the respective areas.

Now, the writers on mensuration have demonstrated, that when two sides of a plane triangle, together with the contained angle are given :—

The area of the triangle is equal to the product of the two sides drawn into half the natural sine of their included angle.

378. This is a principle which we have already stated in the investigation, and expressed analytically in deducing equation (227); we shall now employ it in determining the areas of the triangles according to the magnitudes of the sides and the contained angle, as given in the example and derived from computation.

The natural sine of 38° 43′ is 0.62547, and the sides AC and BC are respectively 28 and 26 inches; consequently, by the above principle, we have

$$a' = \tfrac{1}{2}(28 \times 26 \times 0.62547) = 227.671 \text{ square inches.}$$

The natural sine of the contained angle remaining as above, the sides DC and EC as derived from computation, are equal respectively to 22 and 22.7 inches; hence, from the same principle, we have

$$a'' = \tfrac{1}{2}(22 \times 22.7 \times 0.62547) = 156.1798 \text{ square inches.}$$

Now, according to the conditions of the question, the specific gravity of the fluid is 1000, and that of the floating body is 686; consequently, we obtain

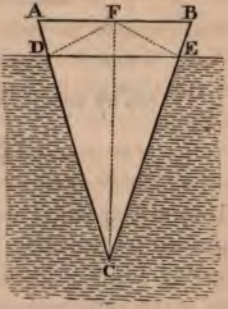
$$1000 : 227.671 :: 686 : 156.1823.$$

In this case the error is extremely small, amounting only to $156.1823 - 156.1798 = 0.0025$ of a square inch; hence we conclude, that the position of equilibrium under the given conditions, is very nearly the same as we have found it to be from the resolution of the equations (227) and (229).

379. The preceding solution, however, indicates only one position of equilibrium; but it is manifest from the nature of the equation (229), that there may be more, for by transposition, we have

$$x^4 - \cos.\phi\sqrt{2(b^2+c^2)-a^2} \times x^3 + \frac{b\,c\,s\cos.\phi'}{s'}\sqrt{2(b^2+c^2)-a^2} \times x - \frac{b^2c^2s^2}{s'^2} = 0,$$

and it is demonstrated by the writers on algebra, that in every equation of an even number of dimensions, having its last term negative, there are at least two real roots, the one positive and the other negative; consequently, the above equation has two of its roots real and determinable; but the equation being of four dimensions has also four roots, hence, the other two roots may also be real, and in that case, there will be three values of x positive and the fourth negative; but for every positive value of x there may be a position of equilibrium, that is, there may be three positions, in which the body may float in equilibrio with the angle ACB downwards; but there cannot be more.



380. If the sides b and c are equal to one another, as represented in the annexed diagram, then $\cos.\phi$ and $\cos.\phi'$ are also equal, and the general equation becomes

$$x^4 - \cos.\phi\sqrt{4b^2 - a^2} \times x^3 + \frac{b^2 s\cos.\phi}{s'}\sqrt{4b^2 - a^2} \times x - \frac{b^4 s^2}{s'^2} = 0. \quad (230).$$

Now, it is manifest from the relation of the terms in this equation, that it is resolveable into the two quadratic factors

$$x^2 - \frac{b^2 s}{s'}, \text{ and } x^2 - \cos.\phi\sqrt{4b^2 - a^2} \times x + \frac{b^2 s}{s'}, \text{ each of which}$$

is equal to nothing; consequently, the four roots of the equation (230), are the same as the roots of the two quadratic equations

$$x^2 = \frac{b^2 s}{s'}, \text{ and } x^2 - \cos.\phi\sqrt{4b^2 - a^2} \times x = -\frac{b^2 s}{s'},$$

and the positions of equilibrium are indicated by the number of real positive roots which these equations contain.

By extracting the square root of both sides of the equation $x^2 = \frac{b^2 s}{s'}$, we shall obtain

$$x = \pm b \sqrt{\frac{s}{s'}}. \qquad (231).$$

This expression exhibits two roots of the original equation (230), one positive and the other negative; but the positive root only becomes available in determining the position of equilibrium, the negative one referring to a case which does not exist.

It has already been shown in equation (227), that when a solid body floats in equilibrio on a fluid of greater specific gravity than itself; then we have

$$x y s' = b c s,$$

but according to the supposition, b and c are equal to one another; hence we get

$$x y s' = b^2 s,$$

from which, by division, we obtain

$$y = \frac{b^2 s}{x s'},$$

or, by substituting the above value of x, it becomes

$$y = \frac{b^2 s}{s'} \div - b \sqrt{\frac{s}{s'}} = \pm b \sqrt{\frac{s}{s'}}. \qquad (232).$$

Hence it appears, that the values of x and y are each of them expressed by the same quantity; consequently, the triangle DCE is isosceles, and AB the extant side of the section, is parallel to DE the base of the immersed portion, both of them being parallel to the plane of floatation or the horizontal surface of the fluid.

381. The practical rule for the reduction of the equation (231) or (232), may be expressed in words at length, in the following manner.

> RULE. *Divide the specific gravity of the solid body, by the specific gravity of the fluid on which it floats; then, multiply the square root of the quotient, by the length of one of the equal sides of the section, and the product will give the portion of that side which is immersed below the plane of floatation, or that which is intercepted between the vertex of the section and the horizontal surface of the fluid.*

382. EXAMPLE. A prism of wood, the sides of whose transverse section are respectively equal to 20, 28 and 28 inches, is placed with its vertex downwards in a cistern or reservoir of water whose

surface is horizontal; it is required to determine, what position the solid will assume when in a state of equilibrium, its specific gravity being to that of water as 686 to 1000?

Here, by the rule, we have

$$\frac{s}{s'} = \frac{686}{1000} \pm 0.686,$$

from which, by extracting the square root, we get

$$\sqrt{0.686} = 0.8282,$$

and finally, by multiplication, we obtain

$$x = 28 \times 0.8282 = 23.1896 \text{ inches.}$$

But according to equation (232), y possesses the very same value; consequently, if 23.1896 inches be set off from the vertex of the section upwards on each of its equal sides, the straight line which joins these points will coincide with the plane of floatation, or the horizontal surface of the fluid on which the body floats.

383. This is the most natural and obvious position of equilibrium, and such as must always obtain when the body is homogeneous, and symmetrical with respect to a vertical plane passing through the axis and bisecting the base; but there may be other situations in which the body may float in a state of quiescence, and the circumstances under which they occur must be determined by the resolution of the following equation, viz.

$$x^2 - \cos.\phi\sqrt{4b^2 - a^2} \times x = -\frac{b^2 s}{s'}.$$

Complete the square, and we shall have

$$x^2 - \cos.\phi\sqrt{4b^2 - a^2} \times x + (\tfrac{1}{2}\cos.\phi\sqrt{4b^2 - a^2})^2 = \tfrac{1}{4}\cos^2.\phi(4b^2 - a^2) - \frac{b^2 s}{s'},$$

and by extracting the square root, we get

$$x - \tfrac{1}{2}\cos.\phi\sqrt{4b^2 - a^2} = \pm\sqrt{\tfrac{1}{4}\cos^2.\phi(4b^2 - a^2) - \frac{b^2 s}{s'}},$$

hence, by transposition, we shall obtain, (233).

$$x = \tfrac{1}{2}\cos.\phi\sqrt{4b^2 - a^2} \pm \sqrt{\tfrac{1}{4}\cos^2.\phi(4b^2 - a^2) - \frac{b^2 s}{s'}},$$

The corresponding values of y are (234).

$$y = \tfrac{1}{2}\cos.\phi\sqrt{4b^2 - a^2} \pm \sqrt{\tfrac{1}{4}\cos^2.\phi(4b^2 - a^2) - \frac{b^2 s}{s'}}.$$

Expressions of this form, arising from the reduction of an adfected quadratic equation, are in general rather troublesome and difficult to render intelligible in words, and even when intelligibly expressed, they are to say the least of them, but very dull and uninviting guides, from which a tasteful reader turns with disgust; we are therefore unwilling to crowd our pages with long and formal directions for the purpose of reducing equations, when it is probable after all, that nine out of every ten of our readers will pass them over, and proceed immediately to discover their object by the direct resolution of the original equation.

384. It is however necessary, in conformity to the plan of our work, to express the most important final equations in words at length, and since the preceding forms are of considerable utility in the doctrine of floatation, it would be a direct violation of systematic arrangement, to omit the verbal description, and leave the subject open only to algebraists; we shall therefore, in order to render both parts of the operation intelligible, endeavour to express the method of reduction in as brief and comprehensive a manner as the nature of the subject will admit.

1. To determine the value of x.

RULE. *From four times the square of one of the equal sides of the section, subtract the square of the base, or side opposite to the vertical angle ; multiply the square root of the remainder by one half the natural cosine of half the vertical angle, and call the product* m.

From four times the square of one of the equal sides of the section, subtract the square of the base, or side opposite to the vertical angle, and multiply the remainder by one fourth of the square of the natural cosine of half the vertical angle, or that which is immersed in the fluid ; then, from the product, subtract the quotient that arises, when the specific gravity of the solid, drawn into the square of one of the equal sides of the section, is divided by the specific gravity of the fluid, and call the square root of the remainder n.

Finally, to and from the quantity denoted by m, *add and subtract the quantity denoted by* n *; then, the sum in the one case, and the difference in the other, will give the two values of* x.

2. To determine the corresponding values of y.

RULE. *Calculate the values of* m *and* n, *precisely after the manner described above ; then, from and to the quantity* *denoted by m, subtract and add the quantity denoted by n, and the difference in the one case, and the sum in the other, will give the values of y corresponding to above values of x.*

385. These are the rules by which the other positions of equilibrium are to be determined; but it is necessary to remark, that beyond certain limits no equilibrium can obtain. In the first place, in order that the body may float with only one of its angles immersed, it is manifestly requisite, that the equal sides of the section should each be greater than $m+n$; and in the second place, in order that x and y may be real positive quantities, the expression $\frac{1}{4}\cos^2.\phi(4b^2-a^2)$ must exceed $\frac{b^2s}{s'}$, or $\frac{s}{s'}$ must be less than $\frac{\cos^2.\phi(4b^2-a\,)}{4b^2}$.

The reason of these limitations is obvious from the nature of the quadratic formula (233) and (234), but it will be more satisfactory to show, that unless the data of the question are so constituted as to fulfil these conditions, the rules will fail in determining the positions of equilibrium; or in other words, there is no other position in which the body will float at rest, but that which is indicated by the equations (231) and (232).

386. Example. The data remaining as in the preceding example, let it be required to determine from thence, whether under the proposed conditions, the body can float at rest in any other position than that which we have already assigned for it, by the reduction of the equations (231) and (232), in which the extant side or base of the figure is parallel to the horizon.

By the principles of Plane Trigonometry, we have

$$\tfrac{1}{2}\cos.\phi=\tfrac{1}{56}\sqrt{28+10)(28-10)}=\tfrac{1}{2}(0.93406)=\tfrac{1}{2}\cos.20^\circ\,55'\,29'';$$

consequently, by proceeding according to the rule, we get

$$m=\tfrac{1}{2}\cos.\phi(4b^2-a^2)^{\frac{1}{2}}=0.46703\sqrt{4\times28^2-20^2}=24.429 \text{ very nearly.}$$

Again, to determine the value of n, it is

$$\tfrac{1}{4}\cos.^2\phi(4b^2-a^2)=0.46703^2(4\times28^2-20^2)=596.768,$$

and for the value of the term, involving the specific gravities, we have

$$\frac{b^2s}{s'}=\frac{28^2\times686}{1000}=537.824;$$

consequently, by subtraction, we get

$$596.768-537.824=58.944.$$

It therefore appears from the last result, that both the values of x and y are real positive quantities; consequently, one of the limiting conditions is answered, and we shall shortly see, whether or not the data are sufficient to satisfy or fulfil the other condition.

By extracting the square root of 58.944, we get

$$n = \sqrt{58.944} = 7.677 \text{ nearly};$$

therefore, by addition and subtraction, the values of x, are

$$x = m + n = 24.429 + 7.677 = 32.106, \text{ and } x = m - n = 24.429 - 7.677 = 16.752 \text{ inches}.$$

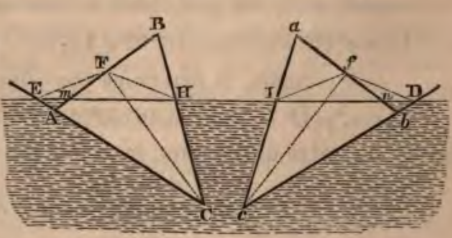
Now, we have seen by equation (234), that the corresponding values of y are expressed in the same terms, having the signs of the second member reversed; hence we have

$$y = 16.752, \text{ and } y = 32.106 \text{ inches}.$$

But here we have $m + n = 32.106$ inches, being greater than b the downward side of the transverse section, which by the question is only 28 inches; it therefore follows, that with the proposed data and under the specified circumstances, there is only one position in which the body can float in a state of rest, and it is that which we have already determined, where the base of the section, or the extant side of the body, is parallel to the surface of the fluid.

But we may here observe, that notwithstanding the values of x and y, as we have just assigned them, do not satisfy the conditions of the question, yet they are not to be considered as being useless; for they actually serve, with a slight modification of the body, to furnish positions in which it will float at rest, although those positions do not agree with the case, in which only one angle of the figure falls below the plane of floatation.

387. The positions of equilibrium corresponding to the preceding values of x and y, are as represented in the annexed diagrams, where ED is the horizontal surface of the fluid, ABC being the position which the body assumes when x is equal to 32.106 and y equal to 16.752 inches, and abc the corresponding position when y is equal to 32.106 and x equal to 16.752 inches; these being the respective values as obtained by the above numerical process.



That the positions here exhibited are those of equilibrium, is very easy to demonstrate, for produce the sides CA and cb to meet the surface of the fluid in the points E and D, and bisect AB and ab in the points F and f; then, if the straight lines FE, FH and fD, fI be drawn, they will be equal among themselves.

This is one of the conditions of equilibrium, as we have already demonstrated in the construction of the original diagram, and the other condition is, that the areas of the immersed figures ECH and DcI, are respectively to the whole areas ABC and abc, as the specific gravity of the solid, is to the specific gravity of the fluid which supports it.

Now, if the first of these conditions obtain, that is, if the straight line FE be equal to FH, and fD equal to fI, then, by the principles of Plane Trigonometry, we shall have

$$\mathrm{EC}^2+\mathrm{CF}^2-2\mathrm{EC.CF}\cos.\mathrm{ECF}=\mathrm{HC}^2+\mathrm{CF}^2-2\mathrm{HC.CF}\cos.\mathrm{FCH};$$

but the angles ECF and FCH are equal to one another, and each of them equal to ϕ; consequently, by substituting the literal representatives, we have

$$x^2+d^2-2dx\cos.\phi=y^2+d^2-2dy\cos.\phi,$$

or by expunging the common term d^2, we get

$$x^2-2dx\cos.\phi=y^2-2dy\cos.\phi,$$

and this, by transposing and collecting the terms, becomes

$$x^2-y^2=2d\cos.\phi(x-y);$$

therefore, if both sides of this equation be divided by the factor $(x-y)$, we shall obtain

$$x+y=2d\cos.\phi.$$

Now, by a previous calculation we found x to be equal to 32.106 inches, y 16.752 inches, d equal to $\sqrt{28^2-10^2}$, and cos. ϕ equal to 0.93406; consequently, by substitution, we have

$$32.106+16.752=2\times0.93406\times6\sqrt{19};$$

hence the equality of the lines FE and FH is manifest.

What we have shown above with respect to the triangle ABC, may also be shown to obtain in the triangle abc, the one being equal and subcontrary to the other; this being the case, it is needless to repeat the process; but we have yet to prove, that the area CEH, is to the whole area ABC, as the specific gravity of the floating body, is to that of the fluid on which it floats.

therefore, by the principles of Plane Trigonometry, we get

AC : AF : : rad. : sin.ACF,

or numerically, we shall obtain

$$28 : 10 :: 1 : \sin.\phi = 0.35714,$$

and we have already found that

$$\cos.\phi = \sqrt{(28+10)(28-10)} \div 28 = 0.93406;$$

but according to the arithmetic of sines, it is

$$\sin.2\phi = 2\sin.\phi\cos.\phi,$$

and by substituting the above numerical values, we get

$$\tfrac{1}{2}\sin.2\phi = 0.35714 \times 0.93406 = 0.33359.$$

Then in the triangle ECH, there are given the two sides EC and HC, respectively equal to 32.106 inches and 16.752 inches, together with half the natural sine of the contained angle; to find the area of the triangle.

Now, by the principles of mensuration, the area of any plane triangle is expressed by half the product of any two of its sides, drawn into the natural sine of the contained angle, hence we get

$$32.106 \times 16.752 \times 0.33359 = 179.417 \text{ square inches.}$$

Again, in the isosceles triangle ABC, there are given the sides AC and BC, respectively equal to 28 inches, and half the natural sine of the contained angle ACB, equal to 0.33359; to find the area.

Here, by the principles of mensuration, we have

$$28 \times 28 \times 0.33359 = 261.53456 \text{ square inches;}$$

then, by the property of floatation, it is

$$1000 : 686 :: 261.53456 : 179.413 \text{ square inches.}$$

388. Since this result agrees so very nearly with that derived from a direct computation of the triangular area, we may reasonably conclude, that the positions exhibited in the diagram are those of equilibrium; it is however necessary to remark, that since the weight of the body remains unaltered in what position soever it may be situated, it does not readily appear in what manner the adequate quantity of fluid is displaced, unless we conceive some physical plane, of sufficient breadth and totally destitute of weight, to be fixed on that edge of the solid which becomes immersed by reason of the change of position that the body is supposed to undergo.

This plane, during the oscillation of the prism, will dislodge the fluid which occupies the space EAm or Dbn, and the weight of this quantity of fluid added to that which is displaced by the quadrilateral figure CJmH or C$bn$I, will be equal to the whole weight of the floating body

389. The above modification, however, does not strictly accord with the conditions of the problem; we must therefore inquire, whether the just principles of equilibrium do not depend upon some other element, such as the specific gravity. Now, we have already stated, that in order to have the values of x and y real positive quantities, it is necessary that

$$\frac{s}{s'} \text{ should be less than } \frac{\cos^2.\phi(4b^2-a^2)}{4b^2},$$

and for a similar reason

$$\frac{s}{s'} \text{ must be greater than } \frac{\cos.\phi\sqrt{4b^2-a^2}}{b}-1.$$

And if the specific gravity of the fluid be denoted by unity, as is the case with water, then the specific gravity of the floating body must lie between the limits

$$\frac{\cos^2.\phi(4b^2-a^2)}{4b^2} \text{ and } \frac{\cos.\phi\sqrt{4b^2-a^2}}{b}-1.$$

The specific gravity of the floating body, as we have proposed it in the question, is 686, that of water being denoted by 1000; consequently, when the specific gravity of water is expressed by unity, that of the solid is 0.686; let us therefore try if this number lies between the above limits; for which purpose, we must substitute 28 for b, 20 for a, and 0.93406 for $\cos.\phi$; then we shall have as follows.

For the greater limit we have $s=\frac{0.93406^2(4\times28^2-20^2)}{4\times28^2}=0.761$ nearly.

It is therefore manifest, that the specific gravity of the floating body, as we have employed it, is less than the greater limit, and consequently properly chosen with regard to it, and we have next to inquire if it exceeds the lesser limit; for which purpose, it is

$$s=\frac{0.93406\sqrt{4\times28^2-20^2}}{28}-=0.745.$$

Here then it is obvious, that the lesser limit exceeds the given specific gravity; and from this we infer, that without the modification specified above, the body will not fulfil the conditions of the problem in any other position than that in which its base is parallel to the surface of the fluid; but if the specific gravity of the floating body fall between the numbers 0.761 and 0.745, all other things remaining, then the prism, besides the situation of equilibrium in which its base is parallel to the surface of the fluid, may have two others, in both of which the conditions of the question will be truly satisfied, for only one angle of the figure will fall below the plane of floatation.

In order therefore to exhibit those positions, we shall suppose the specific gravity of the floating prism to be expressed by 0.753, which is the arithmetical mean between the limits above assigned; then, by operating according to the rules under equations (233) and (234), we shall obtain

$m = \frac{1}{2}\cos.\phi\sqrt{4b^2 - a^2} = 0.46703\sqrt{4\times 28^2 - 20^2} = 24.429$ as formerly computed;

and after a similar manner, we have

$$n = \sqrt{\tfrac{1}{4}\cos.^2\phi(4b^2 - a^2) - \frac{b^2 s}{s'}} =$$

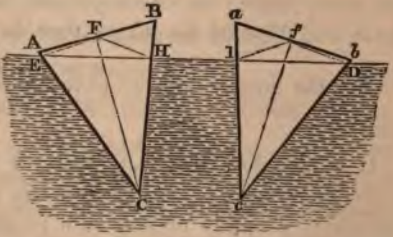
$$\sqrt{0.46703^2(4\times 28^2 - 20^2) - \frac{28^2\times 753}{1000}} = 2.528;$$

consequently, by addition and subtraction, we shall get

$x = m + n = 24.429 + 2.528 = 26.957$ inches, and $x = m - n = 24.429 - 2.528 = 21.901$ inches;

and the corresponding values of y, are
21.901 and 26.957 inches respectively.

390. The positions of equilibrium corresponding to the above values of x and y, are as represented in the annexed diagrams, where it may be shown that the straight lines FE, FH and fD, fI are equal to one another, and also that the areas of the immersed spaces ECH and DcI are respectively to the whole areas ABC and abc, as the specific gravity of the solid, is to that of the fluid on which it floats, or as 0.753 to unity in the case of water.



These conditions being satisfied, the body will float in equilibrio in the positions here exhibited; and it from hence appears, that the problem admits of a complete solution, by retaining the specific gravity of the solid within determinate limits.

391. When the transverse section of the floating prism, is in the form of an equilateral triangle; then a and b are equal to one another, and equation (230) becomes

$$x^4 - b\cos.\phi\sqrt{3}\times x^3 + \frac{b^3 s\cos.\phi\sqrt{3}}{s'}\times x - \frac{b^4 s^2}{s'^2} = 0,$$

and if the value of s' be expressed by unity, as in the case of water, then we have

$$x^4 - b\cos.\phi\sqrt{3}\times x^3 + b^3 s\cos.\phi\sqrt{3}\times x - b^4 s^2 = 0. \quad (235).$$

Now, it is manifest that this equation is composed of the two quadratic factors $x^2 - b^2 s = 0$, and $x^2 - b\cos.\phi\sqrt{3}\times x + b^2 s = 0$, whose roots give the positions of equilibrium.

Since the sides a and b are equal to one another, and s' equal to unity; then, the limits between which the value of s must be retained, are

$$\tfrac{3}{4}\cos^2.\phi \text{ and } \cos.\phi\sqrt{3} - 1;$$

but in the case of the equilateral triangle, $\phi = 30°$; consequently, $\cos.\phi = \frac{1}{2}\sqrt{3}$, and $\cos^2.\phi = \frac{3}{4}$; therefore, by substitution, the above limits become

$$\tfrac{9}{16} = 0.5625, \text{ and } \tfrac{3}{2} - 1 = 0.5,$$

the arithmetical mean of which, is

$$\tfrac{1}{2}(0.5625 + 0.5) = 0.53125.$$

Let this value of s be substituted instead of it, in each of the constituent quadratic factors, and the equations whose roots determine the positions of equilibrium, become respectively

$$x^2 = 0.53125b^2, \text{ and } x^2 - b\cos.\phi\sqrt{3}\times x = -.53125b^2;$$

but by the property of the equilateral triangle,

$$\phi = 30°, \text{ and consequently } \cos.\phi = \tfrac{1}{2}\sqrt{3};$$

hence, the above adfected quadratic equation becomes,

$$x^2 - 1.5bx = -.53125b^2.$$

392. If b the side of the triangle be equal to 28 inches, as we have hitherto supposed it to be; then, the preceding equations become

$$x^2 = 416.5, \text{ and } x^2 - 42x = -416.5.$$

Now, it is manifest, that the first of these equations has one positive and one negative root, each of them being expressed by the same numerical quantity, viz. the square root of 416.5; for by extracting the square root of both sides of the equation, we have

$$x = \pm\sqrt{416.5} = \pm 20.4083 \text{ inches.}$$

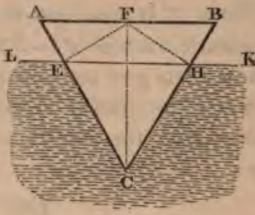
But according to equation (227), we have $xys' = bcs$, where by the present supposition, b and c are equal to one another, and s' is equal to unity; therefore, it is

$$xy = b^2 s = 416.5;$$

hence, by division, we shall get

$$y=\frac{b^2 s}{x}=\frac{416.5}{\pm 20.4083}=\pm 20.4083 \text{ inches.}$$

Then, by taking the positive values of x and y respectively, the position of equilibrium indicated by the above results, is represented in the annexed diagram, where CE and CH are respectively equal to 20.4083 inches, and consequently, AB the base of the section is parallel to LK the surface of the fluid.



Bisect the base AB in the point F, and draw the straight lines CF, FE and FH; then because CE is equal to CH, and the angle ECF equal to the angle HCF, it follows, that the line FE is equal to the line FH; this is one condition that must be satisfied, when the body floats in a state of quiescence; and another is, that the area of the immersed triangle ECH, is to the area of the whole section ACB, as the fraction 0.53125 is to unity.

Now, by the property of the equilateral triangle, the area of the section ACB, is expressed by the product of one fourth of the square of its side, drawn into the square root of the number 3, and the same property holding with respect to the area of the triangle ECH; it follows, that in the case of an equilibrium,

$$\tfrac{1}{4}x^2\sqrt{3} : \tfrac{1}{4}b^2\sqrt{3} : : 0.53125 : 1,$$

or by suppressing the common quantity $\frac{1}{4}\sqrt{3}$, we have

$$x^2 : b^2 : : 0.53125 : 1;$$

but $x^2=416.5$, and $b^2=784$; therefore, by substitution, we obtain

$$416.5 : 784 : : 0.53125 : 1.$$

It is therefore evident, that by the above results, both the conditions of equilibrium are satisfied, and consequently, the body floats in a state of equilibrium when placed as represented in the diagram; that is, with 20.4083 inches of its side immersed, and its base parallel to the plane of floatation.

393. The adfected quadratic equation $x^2-42x=-416.5$, has obviously two positive roots, each of them less than b the side of the section; from which we infer, that besides the position of equilibrium above exhibited, the body may have other two, and these will be determined by the resolution of the equation, as follows.

Complete the square, and we obtain

$$x^2-42x+21^2=-416\ 5+441=24.5,$$

extract the square root of both sides, and we get

$$x-21=\pm\sqrt{24.5}=\pm 4.95 \text{ nearly};$$

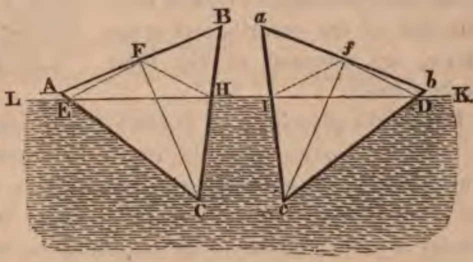
consequently, by transposition, we obtain

$$x=21+4.95=25.95 \text{ inches, and } x=21-4.95=16.05 \text{ inches};$$

and the corresponding values of y are

$$y=\frac{416.5}{25.95}=16.05, \text{ and } y=\frac{416.5}{16.05}=25.95.$$

Now, the positions of equilibrium supplied by the above values of x and y, are as exhibited in the subjoined diagrams, where LK is the surface of the water, ABC the position of the body corresponding to x equal 25.95 inches, and y equal 16.05 inches; abc being the position which the solid assumes, when the values of x and y reverse each other; that is, when x equal 16.05 inches and y equal 25.95 inches.



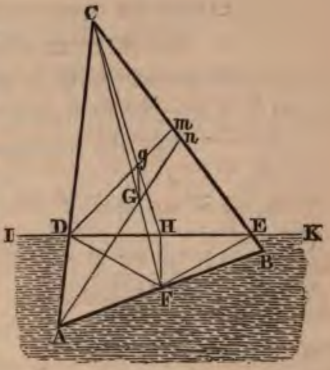
Bisect AB and ab in the points F and f, and draw the straight lines FE, FH and fI, fD to meet the surface of the water in the points E, H and I, D, the points in which the plane of floatation intersects the immersed sides of the solid; then are the lines FE, FH and fI, fD equal among themselves, and the areas ECH, IcD, are respectively to the whole areas ABC, abc as the number 0.53125 to unity.

PROBLEM LVII.

394. Suppose that a solid homogeneous body, in the form of a triangular prism, floats upon the surface of a fluid of greater specific gravity than itself, in such a manner, that two of its edges shall fall below the plane of floatation:—

> *It is required to determine its position, when it has attained a state of perfect quiescence.*

Let ABC represent a section perpendicular to the axis of a solid homogeneous triangular prism, floating in a state of quiescence on a fluid whose horizontal surface is IK; ADEB and DCE being respectively the immersed and extant portions.



Now, it is manifest, that since the whole section ABC, is divided by DE the line of floatation, into the two parts ADEB and DCE; it follows, that the centre of gravity of the section ABC, and the common centre of gravity of the two parts into which it is divided occur in the same point; consequently, the centres of gravity of the triangular areas ABC and DEC, with that of the quadrilateral space ADEB, are situated in the same straight line.

But by the principles of floatation we know, that when the solid is in a state of quiescence, the centre of gravity of the whole section ABC, and that of the immersed portion ADEB occur in the same vertical line; that is, the vertical line passing through their centres of gravity is perpendicular to the horizontal surface of the fluid; and for this reason, the vertical line passing through the centre of gravity of the whole section ABC, and that of the extant portion DEC, is also perpendicular to the horizon.

Bisect the sides AB, BC and DE, EC in the points F, n and H, m and draw the straight lines CF, An and CH, Dm intersecting two and two in the points G and g, which points are respectively the centres of gravity of the triangular spaces ABC and DEC.

Draw the straight lines Gg and FH, and because CF and CH the sides of the triangle CFH, are cut proportionally in the points G and g, it follows, that Gg and FH are parallel to one another; but it has been demonstrated, that Gg is perpendicular to IK the horizontal surface of the fluid; therefore, FH is perpendicular to DE the line of floatation; and since DE is bisected in H, it follows that FD and FE are equal to one another.

Put a = AB, the immersed side of the triangular section ABC,
b = AC, one of the sides of the triangular section which penetrate the fluid,
c = BC, the other penetrating side of the figure,
d = CF, the distance between the vertex of the section and the middle of the immersed side;
ϕ = ACF, the angle contained between the side AC and the bisecting line CF,

$\phi' =$ BCF, the angle contained between the bisecting line CF and the side BC,
$s =$ the specific gravity of the floating solid,
$s' =$ the specific gravity of the supporting fluid,
$x =$ CD, the extant portion of the side AC, and
$y =$ CE, the corresponding portion of the side BC.

Then, since the area of any plane triangle, is expressed by the product of any two of its sides, drawn into half the natural sine of their included angle, it follows, that the area of the entire section ABC, is expressed as under, viz.

$$a' = \tfrac{1}{2} b c \sin.(\phi + \phi'),$$

and for the area of the extant triangle DEC, we have

$$a'' = \tfrac{1}{2} x y \sin.(\phi + \phi'),$$

where the symbols a' and a'', denote the areas of the whole section and the extant portion respectively; consequently, by subtraction, the area of the immersed part ADEB, is

$$(a' - a'') = \tfrac{1}{2}\sin.(\phi + \phi')(bc - xy).$$

But by the principles of floatation, the area of the whole section ABC, is to the area of the immersed portion ADEB, as the specific gravity of the supporting fluid, is to the specific gravity of the floating solid; that is

$$\tfrac{1}{2} b c \sin.(\phi + \phi') : \tfrac{1}{2}\sin.(\phi + \phi')(bc - xy) :: s' : s;$$

from which, by casting out the common terms, we get

$$bc : (bc - xy) :: s' : s,$$

and equating the products of the extremes and means, it is

$$bcs = bcs' - xys';$$

therefore, by transposing and collecting the terms, we obtain

$$xys' = bc(s' - s). \qquad (236).$$

Since the line CF is drawn from the vertex of the triangle ABC, to the middle of the opposite side or base AB, it follows from the principles of geometry, that

$$\text{AC}^2 + \text{BC}^2 = 2(\text{AF}^2 + \text{CF}^2),$$

and this, by substituting the literal representatives, becomes

$$b^2 + c^2 = 2(\tfrac{1}{4}a^2 + d^2);$$

therefore, by transposition, we have

$$4d^2 = 2(b^2 + c^2) - a^2,$$

and finally, by dividing and extracting the square root, we get

$$d = \tfrac{1}{2}\sqrt{2(b^2 + c^2) - a^2}. \qquad (237).$$

This is the very same expression for the value of d as that which we obtained in equation (226), as it manifestly ought to be, since the same letters refer to the same parts of the figure; but we have thought proper to repeat the investigation, in preference to directing the reader's attention to the former result; for by this means, our performance is rendered more systematic, and the several steps of the operation are more readily traced and applied.

Now, in the plane triangle DFC, there are given the two sides CD and CF, with the contained angle DCF; to find the side FD.

Therefore, by the principles of Plane Trigonometry, it is

$$x^2 + d^2 - 2dx \cos.\phi = \mathrm{FD}^2;$$

and in the triangle EFC, there are given the two sides CE and CF, with the contained angle ECF; to find the side FE.

Consequently, as above, we have

$$y^2 + d^2 - 2dy \cos.\phi' = \mathrm{FE}^2;$$

but we have demonstrated, that according to the principles of floatation, the lines FD and FE are equal to one another; therefore, their squares must also be equal; hence, by comparison, we have

$$x^2 - 2dx \cos \phi = y^2 - 2dy \cos.\phi';$$

or by substituting the value of d, equation (237), we get

$$x^2 - \cos.\phi \sqrt{2(b^2+c^2)-a^2} \times x = y^2 - \cos.\phi' \sqrt{2(b^2+c^2)-a^2} \times y. \quad (238).$$

If both sides of equation (236) be divided by the expression xs', we shall obtain

$$y = \frac{bc(s'-s)}{xs'};$$

consequently, by involution, we have

$$y^2 = \frac{b^2c^2(s'-s)^2}{x^2s'^2}.$$

Let these values of y and y^2 be substituted instead of them in equation (238), and we shall have

$$x^2 - \cos.\phi \sqrt{2(b^2+c^2)-a^2} \times x = \frac{b^2c^2(s'-s)^2}{x^2s'^2} - \frac{bc(s'-s)\cos.\phi'}{xs'} \sqrt{2(b^2+c^2)-a^2},$$

and multiplying all the terms by x, we get

$$x^4 - \cos.\phi \sqrt{2(b^2+c^2)-a^2} \times x^3 = \frac{b^2c^2(s'-s)^2}{s'^2} - \frac{bc(s'-s)\cos.\varphi'}{s'}$$
$$\times \sqrt{2(b^2+c^2)-a^2} \times x,$$

and finally, by transposition, we have

$$x^4 - \cos.\phi\sqrt{2(b^2+c^2)-a^2} \times x^3 + \frac{bc(s'-s)\cos.\phi'}{s'}\sqrt{2(b^2+c^2)-a^2}$$
$$\times x = \frac{b^2c^2(s'-s)^2}{s'^2}. \qquad (239).$$

395. The above is the general equation, whose roots give the several positions in which the solid may float in a state of equilibrium; it is similar to equation (229), having $(s'-s)$ instead of s, and $(s'-s)^2$ instead of s^2; the body may therefore have three positions of equilibrium, but it cannot have more, the very same as in the case, where it floated with only one of its edges below the surface of the fluid.

The method of applying the general equation to the determination of the positions of equilibrium, is to calculate the value of d, $\cos.\phi$ and $\cos.\phi'$ from the given dimensions of the section, and to substitute the several given and computed numbers instead of their symbolical equivalents; this will give a numeral equation of the fourth degree, which may be reduced either by approximation or otherwise, according to the fancy of the operator.

396. Example. Suppose a solid homogeneous triangular prism, the sides of whose transverse section are respectively equal to 28, 23 and 18 inches, to float in equilibrio on a cistern of water with two of its edges immersed; it is required to determine the positions of equilibrium, on the supposition that the two longest sides of the section include the extant angle, the specific gravity of the prism being to that of water, as 565 to 1000?

In order to resolve this question, we must first of all determine the length of the line CF, which is drawn from the extant angle at C to the middle of the opposite side AB; for which purpose, let the dimensions of the section be respectively substituted according to the combination exhibited in equation (237), and we shall have

$$d = \tfrac{1}{2}\sqrt{2(28^2+23^2)-18^2} = \tfrac{1}{2}\sqrt{2302} = 23.99 \text{ inches nearly.}$$

Consequently, in the triangles ACF and BCF respectively, we have given the three sides AC, AF, FC and BC, BF, FC to find cos. ACF and cos. BCF; for which purpose, the elements of Plane Trigonometry supply us with the following equations, viz.

In the triangle ACF, it is

$$\cos.\phi = \frac{4(b^2+d^2)-a^2}{8bd},$$

and in the triangle BCF, it is

$$\cos.\phi' = \frac{4(c^2+d^2)-a^2}{8cd}.$$

Therefore, by substituting the respective values of a, b and c, as given in the question, and the value of d as computed above, we shall have the following values of $\cos.\phi$ and $\cos.\phi'$.

Thus, for the absolute numerical value of $\cos.\phi$, it is

$$\cos.\phi=\frac{4(28^2+23.99^2)-18^2}{8\times 28\times 23.99}=0.95166,$$

and for the corresponding value of $\cos.\phi'$, we have

$$\cos.\phi'=\frac{4(23^2+23.99^2-18^2}{8\times 23\times 23.99}=0.92747.$$

Having ascertained the numerical values of $\cos.\phi$ and $\cos.\phi'$, let the respective quantities be substituted in equation (239), and it becomes

$$x^4-0.95166\sqrt{2302}\times x^3+\frac{28\times 23\times 435\times 0.92747}{1000}\sqrt{2302}\times x$$
$$=\frac{28^2\times 23^2\times 435^2}{1000^2},$$

from which, by computing the several terms, we get

$$x^4-45.66x^3+12466x=78478.36.$$

The root of this equation will be most easily discovered by approximation, and for this purpose, we shall adopt the method of trial and error, which Dr. Hutton has so successfully applied to the resolution of every form and order of equations, however complicated may be their arrangement.

By a few simple trials, indeed it is almost self evident, that the value of x will be found between 15 and 16; consequently, by substitution we obtain

$$15^4-45.66\times 15^3+12466\times 15-78478.36=15113.64 \text{ too little.}$$
$$16^4-45.66\times 16^3+12466\times 16-78478.36=509.72 \text{ too great.}$$

Here it is manifest that the errors are of different affections, the one being in defect and the other in excess; hence we have

$$15113.64+509.72 : 16-15 :: 509.72 : 0.032;$$

consequently, for the first approximation, we get

$$x=16-0.032=15.97 \text{ very nearly.}$$

Supposing therefore, that x lies between 15.9 and 16; by repeating the process, we shall have

$$16^4-45.66\times 16^3+12466\times 16-78478.36=509.72 \text{ too great.}$$
$$15.9^4-45.66\times 15.9^3+12466\times 15.9-78478.36=105.393 \text{ too little.}$$

Here again, the errors are of different affections, the one being in excess and the other in defect; consequently, we have

$$509.72 + 105.393 : 16 - 15.9 :: 105.393 : 0.017 \text{ nearly;}$$

therefore, the second approximate value of x, is

$$x = 15.9 + 0.017 = 15.917 \text{ inches.}$$

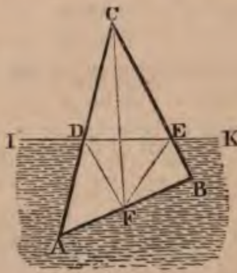
By again repeating the process, a nearer approximation to the true value of x would be obtained, but the above is sufficiently accurate for our present purpose; therefore, let this value of x, together with the numerical values of b, c, s and s', be substituted in equation (236), and we shall obtain

$$15917y = 280140,$$

and from this, by division, we get

$$y = \frac{280140}{15917} = 17.6 \text{ inches.}$$

397. And the position of equilibrium corresponding to the above values of x and y, is represented in the annexed diagram, where IK is the horizontal surface of the fluid, ABED the immersed part of the section, and DCE the extant part.



Bisect AB in F, and draw the straight lines CF, FD and FE; then, as we have previously demonstrated, when the body floats in a state of equilibrium, the lines FD and FE are equal to one another.

Now, in order to determine if this equality obtains, we must have recourse to equation (238), where we have

$$x^2 - \cos.\phi \sqrt{2(b^2 + c^2) - a^2} \times x = y^2 - \cos.\phi' \sqrt{2(b^2 + c^2) - a^2} \times y;$$

then, let the computed values of x, y, $\cos.\phi$, $\cos.\phi'$, and the given values of a, b and c, be substituted instead of them in the above equation, and we shall obtain

$$15.917^2 - 0.95166 \sqrt{2302} \times 15.917 = 17.6^2 - 0.92747 \sqrt{2302} \times 17.6,$$

and this, by transposition and reduction, gives

$$-726.77 = 56.43 - 783.20.$$

398. Another condition of equilibrium is, that the area of the immersed part ABED, is to the area of the whole section ABC, as the specific gravity of the solid is to that of the supporting fluid. This is a more necessary condition than the equality of the lines FD, FE; for such an equality may exist when no equilibrium obtains; but it may be considered as a universal fact, that whenever the two conditions are satisfied at the same time, the body floats in a state of quiescence.

We have already found that $\cos.\phi = 0.95166$, and $\cos.\phi' = 0.92747$; consequently, $\phi = 17° 53'$, and $\phi' = 21° 57'$; hence we have $(\phi + \phi') = 39° 50'$, and by the principles of mensuration, we get

$$a' = \tfrac{1}{2}(28 \times 23)\sin.39° 50' = 322 \times 0.64056 = 206.26032,$$

and the area of the extant part, is

$$a'' = \tfrac{1}{2}(15.917 \times 17.6)\sin.39° 50' = 140.0696 \times 0.64056 = 89.72298;$$

therefore, by subtraction, the area of the immersed part becomes

$$(a' - a'') = 206.26032 - 89.72298 = 116.53734;$$

consequently, by the principle of floatation, it is

$$206.26032 : 116.53734 :: 1000 : 565 \text{ nearly};$$

from which it appears, that both the conditions of equilibrium are satisfied, and therefore the body as exhibited in the figure indicates a state of quiescence.

399. By finding the other roots of the equation, other situations of equilibrium may be assigned; but since the one above given is that which would be adopted in practice, we consider that it would be a waste of both labour and time to search after the others; we therefore leave the reduction of the resulting cubic equation for exercise to the reader, presuming that he will find his trouble and attention amply repaid, by the satisfaction to be derived from the confirmation of the principles by an actual construction.

400. When the triangle ABC becomes isosceles; that is, when the sides b and c are equal to one another; then $\cos.\phi$ and $\cos.\phi'$ are also equal, and the general equation (239), becomes transformed into

$$x^4 - \cos.\phi\sqrt{4b^2 - a^2} \times x^3 + \frac{b^2(s' - s)\cos.\phi}{s'}\sqrt{4b^2 - a^2} \times x = \frac{b^4(s' - s)^2}{s'^2},$$

or by transposing the absolute given quantity $\frac{b^4(s' - s)^2}{s'^2}$, we get

$$x^4 - \cos.\phi\sqrt{4b^2 - a^2} \times x^3 + \frac{b^2(s' - s)\cos.\phi}{s'}\sqrt{4b^2 - a^2} \times x - \frac{b^4(s' - s)^2}{s'^2} = 0. \tag{240}$$

Now, by carefully examining the nature of this equation, it will immediately appear to be composed of the two quadratic factors

$$x^2 - \frac{b^2(s' - s)}{s'} = 0, \text{ and } x^2 - \cos.\sqrt{4b^2 - a^2} \times x + \frac{b^2(s' - s)}{s'} = 0,$$

where it is manifest, that each of these expressions involve two roots of the original equation, and the number of the real positive roots, indicates the number of positions in which the body may float in a state of quiescence, while the absolute values of the roots determine the positions themselves.

401. Let each of the above quadratic factors be transformed into an equation, by transposing the given term $\frac{b^2(s'-s)}{s'}$, and we shall obtain

$$x^2 = \frac{b^2(s'-s)}{s'}, \text{ and } x^2 - \cos.\phi\sqrt{4b^2-a^2}\times x = -\frac{b^2(s'-s)}{s'};$$

and when the value of s', or the specific gravity of the supporting fluid is expressed by unity, as is the case with water; then, we have for the pure quadratic,

$$x^2 = b^2(1-s). \qquad (241).$$

and for the adfected quadratic, it is

$$x^2 - \cos.\phi\sqrt{4b^2-a^2}\times x = -b^2(1-s). \qquad (242).$$

Let the square root of both sides of equation (241) be extracted, and we shall obtain

$$x = b\sqrt{1-s}; \qquad (243).$$

but from equation (236), we have

$$xy = b^2(1-s),$$

and this, by substituting the above value of x, becomes

$$b\sqrt{1-s}\times y = b^2(1-s);$$

hence, by division, we get

$$y = \frac{b(1-s)}{b\sqrt{1-s}} = b\sqrt{1-s}. \qquad (244).$$

Here then it is manifest, that the values of x and y are each expressed by the same quantity; from which we infer, that the solid floats in a state of equilibrium, when the base of the section is parallel to the surface of the fluid; that is, when the extant portion of the section is also isosceles, having its base coincident with the plane of floatation.

402. The practical rule for computing the equation (243) or (244), may be expressed in words at length, as follows.

> Rule. *From unity, or the specific gravity of the fluid, subtract the specific gravity of the floating solid, and multiply the square root of the difference by one of the equal sides of the section, and the product will express the value of x and y.*

403. Example. Suppose the two equal sides of the section to be respectively equal to 28 inches, the base 18 inches, and the specific

gravity of the solid 0.565 as in the preceding example; how much of the equal sides is immersed in the fluid, and how much is extant, the body being in a state of quiescence?

Here, by the rule, we have

$$1-s=1-0.565=0.435,$$

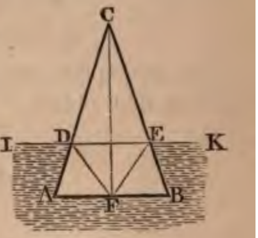
the square root of which is

$$\sqrt{0.435}=0.659;$$

consequently, by multiplication, we have

$$x \text{ or } y=28\times 0.659=18.452 \text{ inches.}$$

404. And the position of equilibrium indicated by the above value of x and y, is as represented in the subjoined diagram, where IK is the horizontal surface of the fluid, ABED the immersed portion of the section, and DCE the extant portion, DE being the water line or plane of floatation.



Since CD and CE are each equal to 18.452 inches, it follows, that AD and BE are each equal to $28-18.452=9.548$ inches, the extant part of the equal sides being nearly double of the immersed part.

Bisect AB in the point F, and draw the straight lines FD and FE; then shall FD and FE be equal to one another; this is manifest, for AF, BF and AD, BE are equal, and the angle DAF is equal to the angle EBF; therefore, FD is equal to FE.

By examining the nature of the equation (243) or (244), it is manifest that the values of x and y depend entirely on the value of s, or the specific gravity of the floating body; now, since this may admit of all magnitudes between zero and unity, which is the specific gravity of water, it follows, that x and y may be of all magnitudes between zero and 28 inches; but whatever may be the magnitude of the extant sides, the position in which the body floats will be the same; viz. that in which the base of the section is parallel to the surface of the fluid.

405. Admitting the specific gravity of the solid to fall within the limits of possibility, the formula equation (242), when reduced, will supply us with other two positions of equilibrium, in which the body may float with two of its angles immersed; here follows the reduction of the equation.

Complete the square, and we obtain

$$x^2-\cos.\phi\sqrt{4b^2-a^2}\times x+\tfrac{1}{4}\cos^2.\phi(4b^2-a^2)=\tfrac{1}{4}\cos^2.\phi(4b^2-a^2)-b^2(1-s),$$

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